1. Find all the co-factors, the adjugate matrix, and the determinant of the matrix
\[
\begin{pmatrix}
1 & 1 & 2 & -1 \\
0 & 2 & -4 & 1 \\
-1 & 0 & 2 & 0 \\
-1 & 0 & 2 & 0 \\
\end{pmatrix}.
\]

2. True or false:
(a) Equivalent matrices have the same determinant; (F)
(b) The determinant of any elementary matrix is 1; (F)
(c) \(\det(A + B) = \det A + \det B\), \(\det(-A) = -\det A\), and \(\det(AB) = (\det A)(\det B)\); (F, F, T)
(d) \(\det A^T = \det A\) and \(\det A^{-1} = (\det A)^{-1}\); (T, T)
(e) A square matrix is invertible if and only if its determinant is nonzero. (T)

3. Calculate the determinant of each of the following matrices:
\[
\begin{pmatrix}
5 & 0 & -1 \\
1 & -3 & -2 \\
0 & 5 & 3 \\
\end{pmatrix}; \quad \begin{pmatrix}
1 & 1 & 2 & -1 \\
0 & 0 & -4 & 0 \\
-1 & 3 & 12 & 0 \\
1 & 2 & -4 & 1 \\
\end{pmatrix}; \quad \begin{pmatrix}
1 & 3 & 3 & -4 \\
0 & 1 & 2 & 5 \\
-1 & 3 & 12 & 0 \\
2 & 5 & 4 & -3 \\
\end{pmatrix}.
\]

4. Show that \(\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} = (a-b)(b-c)(c-a)\). (Hint: transpose, row reduction, and expansion.)

5. Let \(A, B, C\) be three \(2 \times 2\) matrices. Show that \(\det \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} = (\det A)(\det C)\).

6. What is Cramer’s rule? Let \(A\) be a \(3 \times 3\) matrix and assume \(\det A = 2\), the co-factors \(C_{11} = 2\), \(C_{12} = -4\), \(C_{13} = 6\). What is the solution to \(Ax = e_1\) with \(e_1 = (1, 0, 0)^T\)?

7. Since \(A^{-1} = \frac{1}{\det A} \text{adj} A\), we have \(A \text{adj} A = (\det A)I\). Correct? (Yes)

8. Find the area of the parallelogram whose vertices are \((0, -2), (6, -1), (-3, 1), (3, 2)\).

9. Find the area of the parallelepiped whose vertices are \((0, 0, 0), (1, 4, 0), (-1, 2, -1), (-2, -5, 2)\).

10. Show that the area of the triangle with vertices \((x_1, y_1), (x_2, y_2), (x_3, y_3)\) is \(\frac{1}{2} \det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}\).

11. Determine if \(H\) is a subspace of \(\mathbb{R}^3\):
(a) \(H\) consists of all the vectors in \(\mathbb{R}^3\) with the product of all the components equal to 0; (No)
(b) \(H\) consists of all the vectors in \(\mathbb{R}^3\) with the first component equal to 1; (No)
(c) \(H\) consists of all the vectors in \(\mathbb{R}^3\) with the first component equal to 0; (Yes)
(d) \(H\) consists of all the vectors \(\begin{pmatrix} 2a - b \\ 2b - c \\ 2c - a \end{pmatrix}\) with \(a, b, c\) all real numbers. (Yes)

12. Is the set of all \(2 \times 2\) diagonal matrices a subspace of the space of all \(2 \times 2\) matrices? (Yes)
13. Determine if \[
\begin{bmatrix}
-4 \\ -8 \\ 6
\end{bmatrix}
\] is in the span by \[
\begin{bmatrix}
3 \\ 8 \\ 5
\end{bmatrix}, \quad
\begin{bmatrix}
-5 \\ -8 \\ 2
\end{bmatrix}, \quad \text{and} \quad
\begin{bmatrix}
-9 \\ 3 \\ 9
\end{bmatrix}.
\]

14. Determine if \[
\begin{bmatrix} 2 \\ 1 \end{bmatrix}
\] is in the column space and null space of \[
\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}.
\]

15. Let \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) be two vectors in a vector space. Show that any three vectors in \( \text{Span} \{ \mathbf{v}_1, \mathbf{v}_2 \} \) must be linearly dependent.

16. Work out Exercise problems 32 and 33 of Section 4.2 of the textbook.

17. Show that \( p_0(t) = 1, \ p_1(t) = t, \) and \( p_2(t) = 1 + t + t^2 \) form a basis of the vector space \( \mathbb{P}_2 \). Find the coordinates of \( p(t) = 2 - t + 3t^2 \) with respect to this basis.

18. Find a basis for the column space, the null space, and row space of \[
\begin{bmatrix}
1 & -2 \\ -2 & 7 \\ 3 & -9
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 & -1 & 5 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 2 & -1
\end{bmatrix}.
\]

19. True or false:
   (a) Two equivalent matrices have the same null space, column space, and row space; (T, F, T)
   (b) The row space of \( A \) is the column space of \( A^T \); (T)
   (c) If the null space of a square matrix \( A \) is \( \{0\} \) then \( A \) is invertible; (T)
   (d) If \( \mathbf{u} \) and \( \mathbf{v} \) are two vectors in \( \mathbb{R}^3 \) then the rank of the matrix \( \mathbf{uv}^T \) is always 0 or 1. (T)

20. What is the dimension of \( \mathbb{R}^4, \mathbb{P}_4 \), the space of all \( 4 \times 4 \) matrices? (4, 5, 16)

21. If \( V \) is a vector space and \( \text{dim} \ V = n \) then \( n+1 \) vectors in \( V \) must be linearly dependent. Correct? (Yes)

22. True or false:
   (a) \( \text{rank} \ A + \text{rank} \ B = \text{rank} \ (A + B) \); (F)
   (b) \( \text{rank} \ (AB) = (\text{rank} \ A)(\text{rank} \ B) \); (F)
   (c) \( \text{dim} \ \text{Col} \ A + \text{dim} \ \text{Nul} \ A = \text{number of columns of} \ A. \) (T)

23. If \( \lambda \) is an eigenvalue of \( A \). Then \( \lambda^2 \) is an eigenvalue of \( A^2 \). Correct? Why? (Hint: use the definition.)

24. Any three different eigenvectors of a matrix \( A \) corresponding to three different eigenvalues must be linearly independent. Correct? Why? (Yes)

25. Find all the eigenvalues and eigenvectors of \[
\begin{bmatrix}
9 & -2 \\ 2 & -5
\end{bmatrix}
\] and \[
\begin{bmatrix}
6 & -2 & 0 \\ -2 & 9 & 0 \\ 2 & 0 & 2 & -1
\end{bmatrix}.
\]

26. Diagonalize \( A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix} \) and compute \( A^6 \).

27. The matrix \( A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 4 & 1 \\ -1 & -1 & 2 \end{bmatrix} \) has one eigenvalue equal to 2. Diagonalize \( A \).

28. True or false:
   (a) If two matrices have same eigenvalues then either both of them are diagonalizable or both of them are not; (F)
(b) Any invertible matrix is diagonalizable; (F)
(c) Any upper triangular square matrix is diagonalizable; (F)
(d) The inverse of a diagonalizable matrix (if exists) is diagonalizable. (T)