1. Let $X$ be a locally compact Hausdorff (LCH) space, $Y$ a closed subset of $X$ (which is an LCH space in the relative topology), and $\mu$ a Radon measure on $Y$. Define $I : C_c(X) \to \mathbb{C}$ by

$$I(f) = \int (f|_Y) \, d\mu \quad \forall f \in C_c(X),$$

where $f|_Y$ is the restriction of $f$ onto $Y$. Prove that $I$ is a positive linear functional on $C_c(X)$ and that the induced measure $\nu$ on $X$ is given by $\nu(E) = \mu(E \cap Y)$ for any $E \in \mathcal{B}_X$.

2. Let $\mu$ be a $\sigma$-finite Radon measure on an LCH space $X$ and $A \in \mathcal{B}_X$. Prove that the Borel measure $\mu_A$ defined by $\mu_A(E) = \mu(E \cap A)$ ($E \in \mathcal{B}_X$) is a Radon measure on $X$.

3. Let $\mu$ be a Radon measure on an LCH space $X$.

   (1) Let $N$ be the union of all open $U \subseteq X$ such that $\mu(U) = 0$. Prove that $N$ is open and that $\mu(N) = 0$. The complement of $N$ is called the support of $\mu$ and is denoted by supp $(\mu)$.

   (2) Prove that $x \in \text{supp} (\mu)$ if and only if

   $$\int f \, d\mu > 0$$

   for every $f \in C_c(X, [0, 1])$ such that $f(x) > 0$.

4. Let $\mu$ be a Radon measure on an LCH space $X$ and $\phi \in L^1(\mu)$ with $\phi \geq 0$ on $X$. Define

$$\nu(E) = \int_E \phi \, d\mu \quad \forall E \in \mathcal{B}_X.$$

   (1) Prove that $\nu$ is a Radon measure on $X$.

   (2) Prove that supp $(\nu) \subseteq \text{supp} (\phi)$ and give an example to show that the strict inclusion can occur.

5. Let $X$ be the one-point compactification of a set with the discrete topology. If $\mu$ is a Radon measure on $X$, then supp $(\mu)$ is countable.