1. Suppose $X$ follows a Normal distribution with mean $\mu = 2$ and standard deviation $\sigma = 3$.
2. We suppose that the variance is unknown.
3. Data $X_1, \ldots, X_n$: IID

$$X \sim N(\mu, \sigma^2)$$ Confidence interval on $\mu$.

What if $\sigma$ is also unknown? $\hat{\sigma}$

**Estimators:**

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

$$\hat{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \hat{\mu})^2$$

More about this latter.

Recall:

$$\frac{\hat{\mu} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

$\sigma$ being unknown, $\hat{\sigma} \sim \hat{\sigma}$:

$$\frac{\hat{\mu} - \mu}{\hat{\sigma}/\sqrt{n}} \not\sim N(0,1).$$

Instead:

$$\sim \mathcal{N}(n-1)$$

Student $t$ distribution with $n-1$ degrees of freedom.

$$\mathcal{N}(n-1) \rightarrow N(0,1) \text{ as } n \rightarrow +\infty.$$
\[ P\left(-t_{\alpha/2} < \frac{\hat{\mu} - \mu}{\sigma / \sqrt{n}} < t_{\alpha/2}\right) = 1 - \alpha. \]

So CI: \[ [\hat{\mu} - t_{\alpha/2} \frac{\hat{\sigma}}{\sqrt{n}} ; \hat{\mu} + t_{\alpha/2} \frac{\hat{\sigma}}{\sqrt{n}}] \]
5.3.12. During one of the first “beer wars” in the early 1980s, a taste test between Schlitz and Budweiser was the focus of a nationally broadcast TV commercial. One hundred people agreed to drink from two unmarked mugs and indicate which of the two beers they liked better; fifty-four said, “Bud.” Construct and interpret the corresponding 95% confidence interval for \( p \), the true proportion of beer drinkers who preferred Budweiser to Schlitz. How would Budweiser and Schlitz executives each have put these results in the best possible light for their respective companies?

\[ n = 100. \]
\[ X_1, \ldots, X_n \text{ IID.} \]
\[ p = \text{proportion of beer drinkers preferring Bud}. \]
\[ X_i \sim \mathcal{B}(1, p) \]
\[ \sum_{i=1}^{n} X_i \sim \mathcal{B}(n, p). \]

Goal: C.I. on \( p \).

Data: \( \widehat{p} = 54.0\% \) (estimator).

\[ \widehat{p} = \frac{1}{n} \sum_{i=1}^{n} X_i. \]

Distribution q.

Assuming that \( n \) is large: \( \sim \) CLT

\[ E[X_i] = p. \]
\[ \text{Var}(X_i) = p(1-p). \]

CLT:

\[ \frac{\widehat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} \xrightarrow{D} \mathcal{N}(0,1). \]

We show that:

\[ \frac{\widehat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} \xrightarrow{D} \mathcal{N}(0,1). \]
We show that \[ \frac{\hat{p} - p}{\sqrt{\hat{p}(1 - \hat{p})/n}} \xrightarrow{\text{D}} N(0,1) \]

with \( Z \sim N(0,1) \).

\[ P\left( a < \frac{\hat{p} - p}{\sqrt{\hat{p}(1 - \hat{p})/n}} < b \right) \xrightarrow{n \to \infty} P(a < Z < b), \]

where \( a \) and \( b \) are constants.

Hence a CI for \( p \):

\[ \left[ \hat{p} - 3\times12 \frac{\sqrt{\hat{p}(1 - \hat{p})}}{\sqrt{n}} ; \hat{p} + 3\times12 \frac{\sqrt{\hat{p}(1 - \hat{p})}}{\sqrt{n}} \right]. \]

We have \( \hat{p} = 54\% \),

\[ 1 - \alpha = 0.95 \quad \text{and} \quad 3\times12 = 1.96, \]

so \( C.I. = \left[ 0.41, 0.63 \right] \).
1. Suppose $X$ follows a Poisson distribution with parameter $\lambda = 4.5$.
2. Data $X_1, \ldots, X_n$: IID

$$X \sim \mathcal{P}(4.5), \quad E[X] = \lambda, \quad \text{Var}(X) = \lambda.$$ 

**Construct a CI on $\lambda$.**

- **Data $X_1, \ldots, X_n$: IID**.
- **Estimator of $\lambda$:** $$\hat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} X_i.$$ MLE and the moment estimator of $\lambda$.

**CLT:**

$$\frac{\hat{\lambda} - \lambda}{\sqrt{\frac{\lambda}{n}}} \xrightarrow{\mathcal{D}} N(0, 1).$$

We show that $$\frac{\hat{\lambda} - \lambda}{\sqrt{\frac{\lambda}{n}}} \xrightarrow{\mathcal{D}} N(0, 1).$$

When $n$ is large,

$$P\left(-3.841 \leq \frac{\hat{\lambda} - \lambda}{\sqrt{\frac{\lambda}{n}}} \leq 3.841\right) \approx 1 - \alpha.$$

Hence a CI:

$$\left[\hat{\lambda} - \frac{3.841}{\sqrt{\frac{\lambda}{n}}}, \hat{\lambda} + \frac{3.841}{\sqrt{\frac{\lambda}{n}}}\right].$$
\[
\left[ \hat{\theta} - \frac{3\alpha/2}{\hat{\sigma}} \sqrt{\frac{\hat{\sigma}^2}{n}}, \; \hat{\theta} + \frac{3\alpha/2}{\hat{\sigma}} \sqrt{\frac{\hat{\sigma}^2}{n}} \right].
\]

Rk: asymptotic CI
1. Small sample size: $n \leq 30$

Typically, we assume $X_i \sim N(\mu, \sigma^2)$, with $\sigma$ known.

$$CI = \left[ \hat{\mu} - 3\sigma/\sqrt{n} ; \hat{\mu} + 3\sigma/\sqrt{n} \right]. \text{ of level } 1 - \alpha.$$ 

2. Large sample size: $n > 30$

Using the CLT approx.

$$CI \text{ on } \mu : \left[ \hat{\mu} - 3\sigma/\sqrt{n} ; \hat{\mu} + 3\sigma/\sqrt{n} \right].$$

In particular, for a proportion $p : \hat{p} \sim B(n, p)$

$$CI \text{ on } p : \left[ \hat{p} - 3\sqrt{\hat{p}(1-\hat{p})}/\sqrt{n} ; \hat{p} + 3\sqrt{\hat{p}(1-\hat{p})}/\sqrt{n} \right]. \text{ of level } \alpha.$$