6.2.11. As input for a new inflation model, economists predicted that the average cost of a hypothetical “food basket” in east Tennessee in July would be $145.75. The standard deviation ($\sigma$) of basket prices was assumed to be $9.50, a figure that has held fairly constant over the years. To check their prediction, a sample of twenty-five baskets representing different parts of the region were checked in late July, and the average cost was $149.75. Let $\alpha = 0.05$. Is the difference between the economists’ prediction and the sample mean statistically significant?

- Prediction: $145.75$
- Data: $n = 25$ items $x_1, \ldots, x_n$
- Average $\bar{x}_n = 149.75$

This calls for a decision $\left\{ \begin{array}{c} \text{Yes} \\ \text{No} \end{array} \right.$

Hypothesis testing.
What are the assumptions in the problem?

(i) \( \chi_1, \ldots, \chi_n \) : outcomes from IID random variables.

(ii) The variance is supposed known: \( \sigma = 9.5 \).

We add the assumption:

\[
\text{each } \chi_i \sim N(\mu, \sigma^2)
\]

\( \chi_1, \ldots, \chi_n \): empirical average \( \bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \)

\( \bar{X}_n \) is an outcome of \( \bar{X}_n \)
Defining a decision rule

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If the economist prediction is true then: \( \mu = 145.75 \)

This defines a hypothesis:

\[ H_0 : \mu = 145.75 \]  \text{null hypothesis}

\[ H_1 : \mu \neq 145.75 \]  \text{alternative hypothesis}

Test setup:

\[
\begin{cases}
H_0 : \mu = 145.75 \\
H_1 : \mu \neq 145.75
\end{cases}
\]

Decision: from the data \( x_1, \ldots, x_n \).

Null: \( \mu = 145.75 \). \( \sim \) \( \mu_0 \).

Data: \( \bar{x}_n = 149.75 \).

No difference: \( \bar{x}_n - \mu_0 \)
How to choose $t^*$

- To reject $H_0$: $|\bar{x}_n - \mu_0| > t^*$
- If $|\bar{x}_n - \mu_0|$ is large.

No decision rule: reject $H_0$ if $|\bar{x}_n - \mu_0|$ is small. It is true.
Is it possible to always take the correct decision? \[ \text{No!} \]

- There are errors: probabilities.
- \( P(\text{reject } H_0 \mid H_0 \text{ is true}) \): not a conditional probability.
- A decision rule is: reject \( H_0 \) if \( |\bar{X}_n - \mu_0| > t \)

Under \( H_0 \): true mean is \( \mu_0 \).

\[
\bar{X}_n \sim N\left(\mu_0, \frac{\sigma^2}{n}\right).
\]

So, \( P\left(|\bar{X}_n - \mu_0| > t\right) = P\left(\frac{|\bar{X}_n - \mu_0|}{\sigma/\sqrt{n}} > \frac{t}{\sigma/\sqrt{n}}\right) = P\left(|Z| > c\right),
\]

where \( Z \sim N(0,1) \) and \( c = \frac{t}{\sigma/\sqrt{n}} \).

\[
\begin{align*}
\alpha/2 & \quad \quad \alpha/2 \\
\end{align*}
\]

level of significance of the test.

typically small, denoted by \( \alpha \).
• When \( \alpha \) is fixed beforehand: \( \alpha = 0.05 \).
   
   \[ \Rightarrow \text{we must choose } c = 3 \times 1.2 \]

**Decision rule:**

\[ \text{reject } H_0 \text{ if } \left| \frac{\bar{X}_n - \mu_0}{\sigma / \sqrt{n}} \right| > 3 \times 1.2 \text{ at the level } \alpha. \]

\[ \frac{\bar{X}_n - \mu_0}{\sigma / \sqrt{n}} = \frac{149.75 - 145.75}{9.5 / \sqrt{25}} = 2.1 \]

\[ \text{observed } z. \]

If we choose \( \alpha = 0.05 \): then \( 3 \times 1.2 = 1.96 \).

**Conclusion:** we reject \( H_0 \).

**Meaning:** Yes, the difference is statistically significant at the \( \alpha = 0.05 \).

There is evidence in the data against \( H_0 \).
How does the decision depend on the level of significance?

\[ P(|Z| > 2.1) = \text{prob. of observing a value as least as extreme, as the observed value if } H_0 \text{ is true.} \]

\[ = p\text{-value.} \]

**Equivalent decision:** reject \( H_0 \) if \( p\text{-value} < \alpha \)

\[ \text{Here: } p\text{-value} = P(|Z| > 2.1) = 2 \left(1 - P(Z < 2.1)\right) = 2 \left(1 - 0.982\right) = 0.0357. \]

At the level \( \alpha = 0.05 \): we reject \( H_0 \).

At the level \( \alpha = 0.01 \): we accept \( H_0 \).

So the difference is not statistically significant.

So there is no evidence in the data against \( H_0 \).

It does not mean that \( H_0 \) is true!
Decision: reject \( H_0 \) if

\[ \left| \frac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}} \right| > z_{\alpha/2} \]

\( \iff \) reject \( H_0 \) if

\[ \mu_0 > \bar{X}_n + z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \]

\text{or}

\[ \mu_0 < \bar{X}_n - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \]

\( \iff \) reject \( H_0 \) if \( \mu_0 \) does not belong to \( CI_{1-\alpha} \)

with \( CI_{1-\alpha} = \left[ \bar{X}_n - z_{\alpha/2} \frac{\sigma}{\sqrt{n}}, \bar{X}_n + z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \right] \).
6.3.1. Commercial fishermen working certain parts of the Atlantic Ocean sometimes find their efforts hindered by the presence of whales. Ideally, they would like to scare away the whales without frightening the fish. One of the strategies being experimented with is to transmit underwater the sounds of a killer whale. On the fifty-two occasions that technique has been tried, it worked twenty-four times (that is, the whales immediately left the area). Experience has shown, though, that 40% of all whales sighted near fishing boats leave of their own accord, probably just to get away from the noise of the boat.

(a) Let \( p = P \) (Whale leaves area after hearing sounds of killer whale). Test \( H_0: p = 0.40 \) versus \( H_1: p > 0.40 \) at the \( \alpha = 0.05 \) level of significance. Can it be argued on the basis of these data that transmitting underwater predator sounds is an effective technique for clearing fishing waters of unwanted whales?

(b) Calculate the \( P \)-value for these data. For what values of \( \alpha \) would \( H_0 \) be rejected?

\[
\begin{align*}
\hat{p} &= \frac{24}{52} \\
&\approx 0.46
\end{align*}
\]

By default, \( p_0 = 0.4 \).

Is the sound technique efficient?

\[
\begin{cases}
H_0: p = 0.4 \\
H_1: p > 0.4 & \text{one-sided alternative}
\end{cases}
\]

\( P \): the previous test had \( H_1: \mu \neq 145.75 \) two-sided alternative