6.3.1. Commercial fishermen working certain parts of the Atlantic Ocean sometimes find their efforts hindered by the presence of whales. Ideally, they would like to scare away the whales without frightening the fish. One of the strategies being experimented with is to transmit underwater the sounds of a killer whale. On the fifty-two occasions that technique has been tried, it worked twenty-four times (that is, the whales immediately left the area). Experience has shown, though, that 40% of all whales sighted near fishing boats leave of their own accord, probably just to get away from the noise of the boat.

(a) Let \( p = P(\text{Whale leaves area after hearing sounds of killer whale}) \). Test \( H_0: p = 0.40 \) versus \( H_1: p > 0.40 \) at the \( \alpha = 0.05 \) level of significance. Can it be argued on the basis of these data that transmitting underwater predator sounds is an effective technique for clearing fishing waters of unwanted whales?

(b) Calculate the \( P \)-value for these data. For what values of \( \alpha \) would \( H_0 \) be rejected?

\[
\begin{align*}
n &= 52, & 24 \text{ whales left after hearing the sound.} \\
\hat{p} &= \frac{24}{52} \approx 0.46
\end{align*}
\]

By default, \( p_0 = 0.4 \).

Is the sound technique efficient?

\[
\begin{align*}
H_0 & : p = 0.4 \\
H_1 & : p > 0.4 \quad \text{[one-sided alternative]}
\end{align*}
\]

Remark: the previous test had \( H_1 : \mu \neq 145.75 \) [two-sided alternative]
Data \( X_1, \ldots, X_n \): \( \hat{p} = \frac{k}{n} \).

Distribution of \( \hat{p} \) if \( n \) is large: approx. Normal.

Large sample approx.

Under \( H_0 \): true proportion = \( p_0 = 0.40 \).

\[
T = \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} \sim N(0, 1).
\]

\[ \begin{align*}
& H_0: p = p_0 \\
& H_1: p > p_0.
\end{align*} \]

Decision rule: reject \( H_0 \) for large values of \( T \).

Level of significance: \( \alpha \) \( \quad \) \( P(\text{reject } H_0 \mid H_0 \text{ is true}) \)

reject \( H_0 \) if \( \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} > 3\alpha \)

With \( \alpha = 0.05 \): \( 3\alpha = 1.64 \)

\[
0.46 - 0.40 \sim 0.902 \implies \text{accept } H_0.
\]
\[
\frac{0.46 - 0.40}{\sqrt{0.4(1-0.4) / 152}} \approx 0.902 \Rightarrow \text{accept } H_0.
\]

**Conclusion:** the technique is not sufficient.

- **p-value of the test:** 9

\[
P(Z > 0.902) \approx 0.183.
\]

Where \( Z \sim N(0,1) \).
6.2.11. As input for a new inflation model, economists predicted that the average cost of a hypothetical “food basket” in east Tennessee in July would be $145.75. The standard deviation (σ) of basket prices was assumed to be $9.50, a figure that has held fairly constant over the years. To check their prediction, a sample of twenty-five baskets representing different parts of the region were checked in late July, and the average cost was $149.75. Let \( \alpha = 0.05 \). Is the difference between the economists’ prediction and the sample mean statistically significant?

\[
\begin{align*}
H_0 & : \mu = 145.75 \\
H_1 & : \mu \neq 145.75
\end{align*}
\]

\[X_1, \ldots, X_n \text{ i.i.d. from } N(\mu, \sigma^2)\]

\[\sigma = 9.5 \text{ is known.}\]

\[n = 25.\]

\[\text{Level of significance } \alpha :\]

\[P(\text{reject } H_0 \mid H_0 \text{ is true}) = \alpha\]

Type I error

Another error:

\[P(\text{accept } H_0 \mid H_0 \text{ is false})\]

Type II error

\[\mu \text{ is any value } \neq 145.75\]
Type II error

Recall: \[
\frac{X_n - \mu_0}{\sigma / \sqrt{n}} \sim N(0,1) \quad \text{when } H_0 \text{ is true}
\]

Decision rule: reject \( H_0 \) if \[
\left| \frac{X_n - \mu_0}{\sigma / \sqrt{n}} \right| > z_{\alpha/2}.
\]

Suppose that the true mean is \( \mu_1 \neq \mu_0 \).

\[
\frac{X_n - \mu_0}{\sigma / \sqrt{n}} \text{ does not follow an } N(0,1)
\]

We have
\[
\frac{X_n - \mu_0}{\sigma / \sqrt{n}} = \frac{X_n - \mu_1}{\sigma / \sqrt{n}} + \frac{\mu_1 - \mu_0}{\sigma / \sqrt{n}}.
\]

\[
\sim N(0,1) \quad \text{constant}.
\]

\[
\sim N\left(\frac{\mu_1 - \mu_0}{\sigma / \sqrt{n}}, 1\right).
\]

Decision rule: if \( \frac{\mu_1 - \mu_0}{\sigma / \sqrt{n}} > z_{\alpha/2} \), then \( H_0 \) is false.
\[ P(\text{accept } H_0 \mid H_0 \text{ is false}) = \]

Suppose \( \mu_1 = 148.75 \). Then \( \frac{\mu_1 - \mu_0}{\sigma/\sqrt{n}} = 1.657 \)

with \( \alpha = 0.05 \), \( B_{1-12} = 1.96 \)

\[ \sim 0.06189. \text{ Type II error} \]

Suppose \( \mu_1 = 152.75 \)

\[ P(\text{accept } H_0 \mid H_0 \text{ is false} \mid \mu = 152.75) = 0.028. \text{ Type II error} \]
Type II error $P(accept \ H_0 \ | \ \ H_0 \ is \ false)$. 

Power of the test $\beta = 1 - P(accept \ H_0 \ | \ \ H_0 \ is \ false)$. 

Do vary with $\mu$ in $H_1$.

$\mu \mapsto \beta(\mu)$

Power curves
\[ \begin{align*}
H_0 : \; p &= 0.4 \\
H_1 : \; p &> 0.4
\end{align*} \]

Decision rule: reject \( H_0 \) if \[ \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} > z_{\alpha} \]

Suppose the true proportion is \( p_1 = 0.45 \).

In this case:

\[ \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} = \frac{\hat{p} - p_1}{\sqrt{p_1(1-p_1)/n}} + \frac{p_1 - p_0}{\sqrt{p_0(1-p_0)/n}} \]

\[ = \frac{p_1(1-p_1)}{p_0(1-p_0)} \frac{\hat{p} - p_1}{\sqrt{p_1(1-p_1)/n}} + \frac{p_1 - p_0}{\sqrt{p_0(1-p_0)/n}} \]

\[ \sim N(0,1) \]

\[ p_1 = 0.45 \]

\[ \sim N(b, \sigma^2) = N(0.724, 1.015^2) \]

\[ P(\text{accept } H_0 \mid H_0 \text{ false}) = p = p_1 \]
\[ \begin{align*}
\text{Power (at } p = 0.45) &= 1 - 0.81 = 0.19. \\
\end{align*} \]

Remark: In both tests, pointwise power \[ \to 1 \quad \text{as } n \to \infty \]
\[ \to \text{the test is consistent.} \]