Problem 1: Let \( f : A \to B \) and \( g : C \to D \) be functions. Define a new function
\[
f \times g : A \times C \to B \times D
\]
by the formula \((f \times g)(a, c) = (f(a), g(c))\).

1. Prove or disprove: If \( f \) and \( g \) are injective, then \( f \times g \) is injective.
2. Prove or disprove: If \( f \) and \( g \) are surjective, then \( f \times g \) is surjective.

Problem 2: Let \( X \) be a set. For any subset \( A \subseteq X \), the characteristic function \( \chi_A : X \to \{0, 1\} \) is defined by
\[
\chi_A(x) = \begin{cases} 
1 & x \in A \\
0 & x \not\in A.
\end{cases}
\]
Let \( 1 : X \to \{0, 1\} \) be the constant function \( 1(x) = 1 \). Let \( A, B \subseteq X \). Prove the following.

1. We have \( \chi_A \cdot \chi_B = \chi_{A \cap B} \).
2. We have \( 1 - \chi_A = \chi_{A^c} \).
3. We have \( 1 - (1 - \chi_A) \cdot (1 - \chi_B) = \chi_{A \cup B} \).

Problem 3: (You will thank me when you take Math 140 or 142; the intuition comes from the definition.)

Let \( f : \mathbb{R} \to \mathbb{R} \) be a function. Given \( x \in \mathbb{R} \), \( f \) is said to be continuous at \( x \) if for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) so that for any \( y \in \mathbb{R} \) with \( |x - y| < \delta \), we have \( |f(x) - f(y)| < \varepsilon \). The function \( f \) is said to be continuous if it is continuous at every \( x \in \mathbb{R} \).

Define a function \( f : \mathbb{R} \to \mathbb{R} \) by
\[
f(x) = \begin{cases} 
0 & x \leq 0 \\
1 & x > 0.
\end{cases}
\]
Prove or disprove: the function \( f \) is continuous.

Problem 4: (See Problem 3. Again, you will thank me later.)

Let \( f, g : \mathbb{R} \to \mathbb{R} \) be continuous functions. Prove that \( f + g : \mathbb{R} \to \mathbb{R} \) is also continuous, where \( f + g(x) = f(x) + g(x) \).

Hint: You may assume without proof the triangle inequality: For any \( x, y, z \in \mathbb{R} \) we have
\[
|x - z| \leq |x - y| + |y - z|.
\]

Problem 5: Let \( f : X \to Y \) be any function and let \( G_f \subseteq X \times Y \) be the graph of \( f \). Prove that there exists a bijection between \( X \) and \( G_f \).

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1The functions on the left sides of these equations are defined pointwise, i.e., \( \chi_A \cdot \chi_B(x) = \chi_A(x) \cdot \chi_B(x) \) for any \( x \in X \), etc.
Problem 6: Let \( f : X \to Y \) be a function. Prove that \( f \) is surjective if and only if there exists a function \( g : Y \to X \) such that \( f \circ g = I_Y \). Is \( g \) necessarily unique in this case?

Problem 7: As of the beginning of this quarter, UCSD had 2016 math majors. All of these math majors must take an algebra class (Math 100 or Math 103), an analysis class (Math 140 or Math 142) and will certainly take a combinatorics class (Math 154 or Math 184). No student may take both of 100/103, 140/142, or 154/184. \(^2\)

600 of the majors take Math 100. 800 of the majors take Math 140. 700 of the majors take Math 154. 300 majors take both Math 100 and Math 140. 300 majors take both Math 100 and Math 154. 400 majors take both Math 140 and Math 154. 100 majors take all of Math 100, 140, and 154. How many majors take none of Math 100, 140, and 154?

Problem 8: Prove that a composition of surjections is a surjection and that a composition of injections is an injection.

\(^2\)These aren’t quite the math major requirements.