Problem 1: Let $X = \mathbb{Z} \times (\mathbb{Z} - \{0\})$ prove that the binary relation
$$ (a, b) \sim (a', b') \iff ab' = a'b $$
is an equivalence relation on $X$.

Problem 2: Which of the following formulae define a (well-defined) function $f : \mathbb{Q} \to \mathbb{Q}$?
1. $f(a/b) = a^2/b^2$.  
2. $f(a/b) = a^2/b^3$.  
3. $f(a/b) = b/a$.  
4. $f(a/b) = a + b$.  
5. $f(a/b) = (a - b)/(2b)$.

Problem 3: Consider the following binary relations $\sim$ on the following sets $X$. In each case, state whether $\sim$ is reflexive, whether $\sim$ is symmetric, and whether $\sim$ is transitive. For those binary relations which are equivalence relations, describe the equivalence classes.

1. For $X = \mathbb{Z}$, let $a \sim b \iff ab \neq 0$.  
2. For $X = \mathbb{Z}$, let $a \sim b \iff ab \geq 0$.  
3. For $X = \mathbb{Z}^+$, let $a \sim b \iff ab > 0$.  
4. For $X = \mathbb{Z} - \{0\}$, let $a \sim b \iff ab > 0$.  
5. For $X = \mathbb{Z}^+$, let $a \sim b \iff ab < 0$.  
6. For $X = \mathbb{Z} - \{0\}$, let $a \sim b \iff ab < 0$.

Problem 4: ($\mathbb{N} \sim \mathbb{Z}$) Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of non-negative integers and consider the addition binary operation
$$ + : \mathbb{N} \times \mathbb{N} \to \mathbb{N}. $$
You may assume that + satisfies the following axioms on $\mathbb{N}$.
- (Associative) $(a + b) + c = a + (b + c)$ for all $a, b, c \in \mathbb{N}$.  
- (Commutative) $a + b = b + a$ for all $a, b \in \mathbb{N}$.  
- (Zero) $a + 0 = a$ for all $a \in \mathbb{N}$.  
- (Cancellative) For all $a, b, c \in \mathbb{N}$, if $a + c = b + c$, then $a = b$.

Let $X = \mathbb{N} \times \mathbb{N}$ and define the following binary relation $\sim$ on $X$:
$$ (a, b) \sim (c, d) \iff a + d = b + c. $$
Verify that $\sim$ is an equivalence relation. Let $\mathbb{Z}$ denote the set of equivalence classes
$$ \mathbb{Z} = \{ [a, b] : a, b \in \mathbb{N} \}. $$
Define a binary operation $+: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by
$$ [a, b] + [c, d] = [a + c, b + d]. $$
Prove that $+$ is well defined. Also prove that

- $+$ is associative: For all $[a, b], [a', b'], [a'', b''] \in \mathbb{Z}$ we have
  \[(a, b) + [a', b']) + [a'', b''] = (a, b) + (a', b') + [a'', b'']\].

- $+$ is commutative: For all $[a, b], [a', b'] \in \mathbb{Z}$ we have
  \[[a, b] + [a', b'] = [a', b'] + [a, b]\].

- There exists a unique $0 \in \mathbb{Z}$ with the property that
  \[[a, b] + 0 = [a, b]\]
  for all $[a, b] \in \mathbb{Z}$.

- For any $[a, b] \in \mathbb{Z}$, there exists $[c, d] \in \mathbb{Z}$ with the property that
  \[[a, b] + [c, d] = 0\].

**Problem 5:** Suppose that an integer $n$ is a sum of two squares. (i.e., $n = a^2 + b^2$ for some $a, b \in \mathbb{Z}$). Prove that $n = 4q$ or $n = 4q + 1$ or $n = 4q + 2$ for some $q \in \mathbb{Z}$. Deduce that $1234567$ is not a sum of two squares.

**Problem 6:** Use the Euclidean Algorithm to find the greatest common divisor $d$ of $a = 165$ and $b = 252$. Express $d$ as an integer linear combination of $a$ and $b$.

**Problem 7:** How many equivalence relations are there on a set of cardinality 5?