Math 109: Spring 2016  
Midterm 2

Instructions: Please write your name and section number on your blue book. Make it clear in your blue book what problem you are working on. Write legibly and explain your reasoning. This exam is graded out of 100 points. Following these instructions is worth 5 points.

Problem 1: [15] Let $f : X \to Y$ be a function between sets. Suppose there exists a function $g : Y \to X$ such that $g \circ f = I_X$. Prove that $f$ is injective.

Problem 2: [15] Let $X$ be a finite set and $n \in \mathbb{Z}_{\geq 0}$. (a) Carefully state what it means to say “$|X| = n$”. (Hint: Your answer should involve a function.) (b) If $X$ and $Y$ are finite sets, $|X| = n$, and $\varphi : X \to Y$ is a bijection, prove that $|Y| = n$.

Problem 3: [15] Prove or disprove: For any finite sets $X$ and $Y$ we have $|\mathcal{P}(X \times Y)| = |\mathcal{P}(X) \times \mathcal{P}(Y)|$.

Problem 4: [15] Observe that we can write the set $\mathbb{Q}$ of rational numbers as

$$\mathbb{Q} = \bigcup_{n \geq 1} \{-n, -n + \frac{1}{n}, -n + \frac{2}{n}, \ldots, n - \frac{1}{n}, n\}.$$  

Use this observation to prove that $\mathbb{Q}$ is countable.

Problem 5: [15] Let $S$ be a sphere and let $P_1, P_2, P_3, P_4, P_5$ be five points on $S$. Prove that there is a closed hemisphere $H$ on $S$ containing at least four of these points.

Problem 6: [20] Is $\text{Fun}(\mathbb{Z}^+, \{0, 1\})$ a countable set? Justify your answer.