Problem 1: Prove that for all real numbers $a, b, c, d$ we have
\[-2ab - 2ac - 2ad - 2bc - 2bd - 2cd \leq a^2 + b^2 + c^2 + d^2.\]

Solution: Let $a, b, c, d$ be real numbers. We have that $0 \leq (a + b + c + d)^2$. Expanding yields $0 \leq a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd$. By the Addition Law, we have $-2ab - 2ac - 2ad - 2bc - 2bd - 2cd \leq a^2 + b^2 + c^2 + d^2$, as desired.

Problem 2: Let $a$ be an integer. Prove that 0 divides $a$ if and only if $a = 0$. (Hint: This problem has two parts!)

Solution: Suppose that 0 divides $a$. By the definition of ‘divides’, there is an integer $q$ such that $a = 0q$, so that $a = 0q = 0$, as desired.

Suppose that $a = 0$. Then we have that $a = 0(1)$, so that 0 divides $a$, as desired.

Problem 3: Let $a, b,$ and $c$ be integers. Prove that if $a$ divides $b$ or $a$ divides $c$, then $a$ divides $bc$.

Solution: Suppose $a$ divides $b$. Then there exists an integer $q$ such that $b = aq$, so that $bc = a(qc)$ and $a$ divides $bc$, as desired.

Suppose that $a$ divides $c$. Then there exists an integer $k$ such that $c = ak$, so that $bc = a(kb)$ and $a$ divides $bc$, as desired.

Problem 4: For this problem, we define an integer $n$ to be ‘odd’ if there is another integer $q$ such that $n = 2q + 1$. We define an integer $n$ to be ‘even’ if $n$ is not odd.

Prove that if $n$ is an integer and $n^2$ is even, then $n$ is even.

Solution: Let $n$ be an integer. We will prove the contrapositive statement “if $n$ is odd, then $n^2$ is odd”.

Assume that $n$ is odd. By definition, this means there is an integer $q$ such that $n = 2q + 1$. So, $n^2 = (2q + 1)^2 = 4q^2 + 4q + 1$ is also odd, as desired.

Problem 5: What is wrong with the following “proof” that 1 is the largest integer?

“Let $n$ be the largest integer. Then, since 1 is an integer we must have $1 \leq n$. On the other hand, since $n^2$ is also an integer we must have $n^2 \leq n$ from which it follows that $n \leq 1$ (since $n$ is positive). Thus, since $1 \leq n$ and $n \leq 1$ we must have that $n = 1$. Thus 1 is the largest integer as claimed.”

What does this argument prove?

Solution: The first sentence of the argument presented here presupposes the existence of a largest integer. On the other hand, if the first sentence were changed to “Suppose for contradiction that there was a largest integer; call it $n$.” and if the last sentence were changed to “But since (for example) 1 < 2, 1 is certainly not the largest integer, which is a contradiction.”, this would be a valid proof by contradiction that there is no largest integer.
Problem 6: Recall from class (or from the textbook) that we have the following axioms concerning inequality of real numbers.

1. (Trichotomy Law) For any two real number $a$ and $b$, one and only one of $a < b$, $a = b$, or $a > b$ holds.
2. (Addition Law) For any three real numbers $a$, $b$, and $c$, we have that $a < b$ if and only if $a + c < b + c$.
3. (Multiplication Law) For any three real numbers $a$, $b$, and $c$, we have that ($a < b$ if and only if $ac < bc$, if $c > 0$) and ($a < b$ if and only if $ac > bc$, if $c < 0$).
4. (Transitive Law) For any three real number $a$, $b$, and $c$, if $a < b$ and $b < c$, then $a < c$.

Use these axioms to prove by contradiction that there is no smallest positive real number.

Solution: Suppose for contradiction that there was a smallest positive real number; call it $x$. We have that $0 < x$ and since $\frac{1}{2}$ is positive the Multiplication Law implies that $0 = 0(\frac{1}{2}) < \frac{x}{2}$, so that $\frac{x}{2}$ is a positive real number. The Addition Law implies that $\frac{x}{2} = 0 + \frac{x}{2} < \frac{x}{2} + \frac{x}{2} = x$, so that $\frac{x}{2} < x$. This contradicts the assumption that $x$ is the smallest positive real number.