1. We induct on \( n \). If \( n = 1 \), the formula reduces to \( |A_1| = |A_1| \), so the claim is true in this case.

Let \( k \in \mathbb{Z}^+ \) and suppose that the result holds for \( n = k \). We have

\[
|A_1 \cup A_2 \cup \cdots \cup A_k \cup A_{k+1}| = |A_1 \cup \cdots \cup A_k| + |A_{k+1}| - |(A_1 \cup \cdots \cup A_k) \cap A_{k+1}|
\]

\[\begin{align*}
&= \sum_{\emptyset \neq K \subseteq \mathbb{N}_k} (-1)^{|K|-1}|A_K| + |A_{k+1}| - |(A_1 \cap A_{k+1}) \cup \cdots \cup (A_k \cap A_{k+1})| \\
&= \sum_{\emptyset \neq K \subseteq \mathbb{N}_k} (-1)^{|K|-1}|A_K| + |A_{k+1}| - \sum_{\emptyset \neq J \subseteq \mathbb{N}_k} (-1)^{|J|-1}(-1)^{|J|-1}|A_J \cap A_{k+1}|
\end{align*}\]

(3)

\[\begin{align*}
&= \sum_{\emptyset \neq J \subseteq \mathbb{N}_{k+1}} (-1)^{|J|-1}|A_J|
\end{align*}\]

(4)

where the first equality used the Principle of Inclusion-Exclusion for two sets, the second equality applied the inductive hypothesis to the first term and used DeMorgan’s Law on the third, the third equality used the inductive hypothesis on the third term, and the fourth equality is bookkeeping.

By induction, we conclude that the claimed result holds for any positive integer \( n \).

2. Define \( f : A \to \{1, 3, 5, \ldots, 2n-1\} \) by letting \( f(a) \) equal the largest odd integer dividing \( a \) for \( a \in A \). Since \( |A| = n + 1 \) and \( \{1, 3, 5, \ldots, 2n-1\} = n \), the Pigeonhole Principle implies that there exist \( a_1, a_2 \in A \) with \( a_1 < a_2 \) such that \( f(a_1) = f(a_2) \). Let \( s \) denote the common value of \( f(a_1) \) and \( f(a_2) \). We have that there exist non-negative integers \( r < t \) such that \( a_1 = 2^r s \) and \( a_2 = 2^t s \). This means \( a_2 \divides 2^{t-r} a_1 \), so that \( a_1 \) divides \( a_2 \).

3. Let \( X \) be a finite non-empty set of real numbers with \( |X| = n \). We induct on \( n \). If \( n = 1 \), then \( X = \{x\} \) for some \( x \in \mathbb{R} \) and \( x = \min(X) \).

Let \( k \geq 1 \) and assume the result holds for \( n = k \). If \( |X| = k + 1 \), then write \( X = \{x_1, x_2, \ldots, x_k, x_{k+1}\} \). By the inductive hypothesis, the set \( \{x_1, \ldots, x_k\} \) contains a minimum element; call it \( x \). If \( x \leq x_{k+1} \), then \( x = \min(X) \). If \( x > x_{k+1} \), then \( x_{k+1} = \min(X) \). In either case, \( X \) has a minimum element. The result follows by induction.

4. For all \( b \in B \), \( \min(B) \leq b \). In particular, \( \min(B) \leq a \) for all \( a \in A \). Since \( \min(A) \in A \), we have \( \min(B) \leq \min(A) \).

We have that \( \min(A) \leq a \) for all \( a \in A \). Since \( \max(A) \in A \), we have that \( \min(A) \leq \max(A) \).

We have that \( \max(B) \geq b \) for all \( b \in B \). In particular, \( \max(B) \geq a \) for all \( a \in A \). In particular, since \( \max(A) \in A \), we have \( \max(A) \leq \max(B) \).

5. Define a function \( g : \mathcal{P}_k(X \cup Y) \to \bigcup_{i=0}^k \mathcal{P}_i(X) \times \mathcal{P}_{k-i}(Y) \) by \( g(C) = (C \cap X, C \cap Y) \). For \( C \in \mathcal{P}_k(X \cup Y) \), we have \( f(g(C)) = f(C \cap X, C \cap Y) = (C \cap X) \cup (C \cap Y) = C \), so \( f \circ g \) is the identity function on \( \mathcal{P}_k(X \cup Y) \). Moreover, if \( 0 \leq i \leq k \) and \( (A, B) \in \mathcal{P}_i(X) \times \mathcal{P}_{k-i}(Y) \), then \( f((A, B)) = (A \cap X, B \cap Y) \in \mathcal{P}_i(X) \times \mathcal{P}_{k-i}(Y) \).
then \(g(f(A, B)) = g(A \cup B) = (A \cap X, B \cap Y) = (A, B)\), so \(g \circ f\) is the identity function on \(\bigcup_{i=0}^{k} \mathcal{P}_i(X) \times \mathcal{P}_{k-i}(Y)\). It follows that \(g = f^{-1}\) and \(f\) is a bijection.

Let \(X\) and \(Y\) be disjoint sets with \(|X| = n\) and \(|Y| = m\). Then for any positive integer \(k\), we have that \(|\mathcal{P}_k(X \cup Y)| = \binom{n+m}{k}\) (where we used the fact that \(|X \cup Y| = n + m\) because \(X\) and \(Y\) are disjoint). On the other hand, the cardinality of \(\bigcup_{i=0}^{k} \mathcal{P}_i(X) \times \mathcal{P}_{k-i}(Y)\) is \(\sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i}\). The fact that the given function \(f\) is a bijection implies that these two cardinalities are equal.

6. Define a function \(\phi : \{(A, B) : A \subseteq B \subseteq \mathbb{N}_n\} \rightarrow \text{Fun}(\mathbb{N}_n, \mathbb{N}_3)\) by the formula

\[
\phi(A, B)(i) = \begin{cases} 
1 & i \in A \\
2 & i \in B - A \\
3 & i \notin B 
\end{cases}
\]

for \(i \in \mathbb{N}_n\). If we can show that \(\phi\) is a bijection we are done because \(|\text{Fun}(\mathbb{N}_n, \mathbb{N}_3)| = 3^n\) by a result we proved in class. To see that \(\phi\) is a bijection, define a function \(\psi : \text{Fun}(\mathbb{N}_n, \mathbb{N}_3) \rightarrow \{(A, B) : A \subseteq B \subseteq \mathbb{N}_n\}\) by \(\psi(f) = (f^{-1}(\{1\}), f^{-1}(\{1, 2\}))\). It is routine to check that \(\psi = \phi^{-1}\).