Math 190: Fall 2014
Homework 7 Solutions
Due 5:00pm on Friday 12/5/2014

Problem 1: Let $I = [0, 1]$. Prove that there is no continuous bijection $f : I \to I^2$. (On the other hand, there is a continuous surjection $I \to I^2$.)

Solution: Suppose $f : I \to I^2$ is a continuous bijection. Then $I$ is compact and $I^2$ is Hausdorff, so $f$ is a homeomorphism. The restriction $f|_{I-\{1/2\}} : I - \{1/2\} \to I^2 - \{f(1/2)\}$ is also a homeomorphism. But $I - \{1/2\}$ is not connected and $I^2 - \{f(1/2)\}$ is connected.

Problem 2: Let $(X, <)$ be an ordered set in the order topology. Suppose that for all $a < b$ in $X$, the closed interval $[a, b]$ is compact. Prove that $X$ has the least upper bound property: if $A$ is any nonempty subset of $X$ which has an upper bound, then $A$ has a least upper bound.

Solution: Suppose $X$ does not have the least upper bound property. Then there exists a nonempty subset $A \subset X$ which has an upper bound and no least upper bound. Let $a_0 \in A$ and let $b_0$ be an upper bound of $A$. The open sets $\{(a, \infty) : a \in A\}$ cover $\{x \in X : x$ is not an upper bound of $A\}$. On the other hand, since $A$ has no least upper bound, the open sets $\{(b, \infty) : b$ is an upper bound of $A\}$ cover $\{x \in X : x$ is an upper bound of $A\}$. Therefore, we have an open cover $\mathcal{U}$ of $X$ (and hence of $[a_0, b_0]$) given by

$$\mathcal{U} := \{(-\infty, a) : a \in A\} \cup \{(b, \infty) : b$ is an upper bound of $A\}.$$

We claim that no finite subset of $\mathcal{U}$ covers $[a_0, b_0]$. For if it did, then there would exist elements $a_1, \ldots, a_n \in A$ and upper bounds $b_1, \ldots, b_m$ of $A$ such that $[a_0, b_0] \subset (-\infty, a_1) \cup \cdots \cup (-\infty, a_n) \cup (b_1, \infty) \cup \cdots \cup (b_m, \infty)$. Letting $a = \max(a_1, \ldots, a_n)$ and $b = \min(b_1, \ldots, b_m)$, we have $[a_0, b_0] \subset (a, \infty) \cup (b, \infty)$. However, since $b$ is an upper bound of $A$ we know that $A \cap (b, \infty) = \emptyset$, so that $a_0 \not\in (a, \infty)$. Since $b_0$ is an upper bound of $A$ we know that $a \in [a_0, b_0]$. But $a \not\in (b, \infty)$ (because $a \in A$) and $a \not\in (-\infty, a)$. This is a contradiction.

Problem 3: Let $X$ be a compact Hausdorff space. Prove that $X$ is normal: One-point sets in $X$ are closed and for any disjoint closed sets $A, B \subset X$, there exist disjoint open sets $U, V \subset X$ such that $A \subset U$ and $B \subset V$.

Solution: One-point sets in $X$ are closed because $X$ is Hausdorff. Let $A$ and $B$ be disjoint closed subsets of $X$. For any $a \in A$, we proved in class that we can find neighborhoods $U_a$ of $a$ and $V_a$ of $B$ such that $U_a \cap V_a = \emptyset$. Then $\{U_a : a \in A\}$ is an open cover of $A$. Since $X$ is compact and $A$ is closed, $A$ is also compact. So there exist $a_1, \ldots, a_n \in A$ such that $A \subset U := U_{a_1} \cup \cdots \cup U_{a_n}$. Then $V := V_{a_1} \cap \cdots \cap V_{a_n}$ is a neighborhood of $B$ and $U \cap V = \emptyset$.

Problem 4: (Exercise 28.1 in Munkres) Prove that $[0, 1]$ is not limit point compact as a subspace of $\mathbb{R}_d$. 

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Solution: Consider the infinite set \( A = \left\{ 1 - \frac{1}{n} : n = 1, 2, \ldots \right\} \). We claim that \( A \) has no limit point. Indeed, \( 0 \) is not a limit point of \( A \) because \( [0,1/2) \) is an open neighborhood of \( 0 \) which intersects no point of \( A \). For \( 0 < x < 1 \), \( x \) is not a limit point of \( A \) because \( \left( \frac{x-1}{n}, \frac{x}{n+1} \right) \) is an open neighborhood of \( x \) which intersects no point in \( A \) other than \( x \) for an appropriate value of \( n \). Finally, \( 1 \) is not a limit point of \( A \) because \( \{1\} \) is an open neighborhood of 1 which intersects no point of \( A \).

Problem 5: (Exercise 28.6 in Munkres) Let \((X, d)\) be a metric space and let \( f : X \to X \) be an isometry. For all \( x, y \in X \) we have \( d(x, y) = d(f(x), f(y)) \). Suppose \( X \) is compact. Prove that \( f \) is bijective and hence a homeomorphism. (Hint: If \( a \notin f(X) \), choose \( \epsilon > 0 \) such that \( B_d(a, \epsilon) \) does not intersect \( f(X) \). Let \( x_1 = a \) and let \( x_{n+1} = f(x_n) \) in general. Prove that \( d(x_m, x_n) \geq \epsilon \) for all \( m \neq n \).)

Solution: Since \( X \) is compact, we know that \( f(X) \subseteq X \) is compact. Since \( X \) is a metric space, and hence Hausdorff, we know that \( f(X) \) is closed in \( X \). If \( a \in X - f(X) \), we can find \( \epsilon > 0 \) such that \( B_d(a, \epsilon) \) does not intersect \( f(X) \). Define a sequence \((x_n)_{n \geq 1}\) by \( x_1 = a \) and \( x_{n+1} = f(x_n) \) for \( n \geq 1 \). We claim that \( d(x_m, x_n) \geq \epsilon \) for all \( m \neq n \). To see this, observe that \( x_n \in f(X) \) for \( n \geq 2 \), so that \( d(x_1, x_n) \geq \epsilon \) for all \( n \geq 2 \). Since \( f \) is an isometry, we have that \( d(x_k, x_{n+k}) = d(x_1, x_n) \) for all positive integers \( k \) and \( n \). The claim follows and we conclude that \((x_n)_{n \geq 1}\) has no convergent subsequence. But \( X \) is a compact metric space, and hence sequentially compact. This is a contradiction.

Problem 6: Are continuous images of limit point compact spaces necessarily limit point compact? Are closed subsets of limit point compact spaces necessarily limit point compact?

Solution: The first answer is no. Let \( Y = \{a, b\} \) be a two-point set with the indiscrete topology and endow the space \( X := Y \times \mathbb{Z}_{>0} \) with the product topology. We saw in class that \( X \) is limit point compact. However, the map \( f : X \to \mathbb{Z}_{>0} \) given by \( f(a, n) = f(b, n) = n \) is a continuous surjection and \( \mathbb{Z}_{>0} \) is not limit point compact (indeed, \( \mathbb{Z}_{>0} \) itself is an infinite subset without a limit point). The second answer is yes. Let \( X \) be a limit point compact and let \( Y \subseteq X \) be closed. If \( A \subseteq Y \) is infinite, then \( A \) has a limit point in \( X \), say \( x \). But \( x \) is also a limit point of \( Y \) in \( X \), and \( Y \) contains all of its limit points (being closed). This means \( x \in Y \), so that \( Y \) is limit point compact.

Problem 7: Let \( X \) be a locally compact space and let \( f : X \to Y \) be a continuous surjection. Is \( Y \) necessarily locally compact?

Solution: No. Let \( \mathbb{Q}_d \) denote the set \( \mathbb{Q} \) of rational numbers, endowed with the discrete topology. Then the identity function \( i : \mathbb{Q}_d \to \mathbb{Q} \) is a continuous surjection (where we give the codomain the standard topology). \( \mathbb{Q}_d \) is discrete, and hence locally compact. However, \( \mathbb{Q} \) is not locally compact.

Problem 8: (Exercise 29.6 in Munkres) Prove that the one-point compactification of \( \mathbb{R} \) is homeomorphic to the circle \( S^1 \).
Solution: We use the characterization of the one-point compactification given in Munkres. Let $N = (0, 1) \in S^1$ denote the “north pole”. Then $S^1 - \{N\}$ (in the subspace topology inherited from $S^1$) is homeomorphic to $\mathbb{R}$ and $S^1$ is compact. By the uniqueness of the one-point compactification, this forces $S^1$ to be the one-point compactification of $\mathbb{R}$.

Problem 9: Let $M_1$ and $M_2$ denote two copies of the Möbius band. Let $\sim$ denote the equivalence relation on $M_1 \amalg M_2$ which identifies a point on the boundary circle of $M_1$ with the corresponding point on the boundary circle $M_2$. Prove that the quotient space $(M_1 \amalg M_2)/\sim$ is homeomorphic to the Klein bottle $K$.

Solution: We can identify $K$ with the quotient $I^2/\sim$, where $(x, 0) \sim (x, 1)$ for $0 \leq x \leq 1$ and $(0, y) \sim (1, 1 - y)$ for $0 \leq y \leq 1$. We define two subspaces $A_1$ and $A_2$ of $I^2$ by
\[
A_1 := \{(x, y) \in I^2 : 1 \leq x + 2y \leq 2\},
\]
\[
A_2 := \{(x, y) \in I^2 : x + 2y \leq 1 \text{ or } 2 \leq x + 2y\}.
\]
Then $A_1$ and $A_2$ are two closed subspaces of $I^2$ satisfying $A_1 \cup A_2 = I^2$. Moreover, we have that $A_1$ and $A_2$ are saturated with respect to $\sim$ in the sense that if $(x, y) \sim (x', y')$ and $(x, y) \in A_i$ (for $i = 1, 2$), then $(x', y') \in A_i$. Therefore, we can consider $\sim$ as an equivalence relation on either $A_1$ or $A_2$. We have that $A_1/\sim$ and $A_2/\sim$ are both homeomorphic to the Möbius band. The quotient of the intersection $(A_1 \cap A_2)/\sim$ is homeomorphic to the circle, and forms the boundaries of the Möbius bands $A_1/\sim$ and $A_2/\sim$. 