Problem 1: Let $X$ be a set, let $\mathcal{B}$ be a basis for a topology on $X$, and let $\mathcal{T}$ be the topology generated by $\mathcal{B}$. Prove that $\mathcal{T}$ equals the intersection of all topologies on $X$ containing $\mathcal{B}$.

Problem 2: Consider the following five topologies on $\mathbb{R}$:

- $\mathcal{T}_1$ = the standard topology,
- $\mathcal{T}_2$ = the topology $\mathbb{R}_K$,
- $\mathcal{T}_3$ = the finite complement topology,
- $\mathcal{T}_4$ = the upper limit topology, having all sets $(a, b]$ as a basis,
- $\mathcal{T}_5$ = the topology having all sets $(-\infty, a]$ as a basis.

Determine, for each of these topologies, which of the others it contains.

Problem 3: We proved in class that the collection of rational open intervals $\{(a, b) : a, b \in \mathbb{Q}\}$ is a basis for the standard topology on $\mathbb{R}$. Is the set of rational half-open intervals $\{[a, b) : a, b \in \mathbb{Q}\}$ a basis for the lower limit topology $\mathbb{R}_\ell$ on $\mathbb{R}$?

Problem 4: Let $S_\Omega$ be the minimal uncountable well ordered set.

1. Prove that $S_\Omega$ does not have a largest element.
2. Prove that for any $\alpha \in S_\Omega$, the set $\{x \in S_\Omega : x > \alpha\}$ is uncountable.
3. Let $X_0$ be the set of all elements of $S_\Omega$ which do not have an immediate predecessor. Prove that $X_0$ is uncountable. 1

Problem 5: Let $X$ be a topological space and suppose $A \subset Y \subset X$. Give $Y$ the subspace topology. Prove that the topology that $A$ inherits as a subspace of $X$ equals the topology that $A$ inherits as a subspace of $Y$.

Problem 6: Give $\mathbb{R}$ the standard topology and consider $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following five sets are open in $Y$? Which are open in $\mathbb{R}$?

- $A = \{x : \frac{1}{2} < x < 1\}$.
- $B = \{x : \frac{1}{2} < x \leq 1\}$.
- $C = \{x : \frac{1}{2} \leq x < 1\}$.
- $D = \{x : \frac{1}{2} \leq x \leq 1\}$.
- $E = \{x : 0 < |x| < 1 \text{ and } 1/x \notin \mathbb{Z}_{>0}\}$.

Problem 7: Endow $\mathbb{R}^n$ with the standard topology. Prove that this topology has a countable basis.

Problem 8: Let $L$ be a straight line in the plane. Describe the topology that $L$ inherits as a subspace of $\mathbb{R}_\ell \times \mathbb{R}$ and the topology that $L$ inherits as a subspace of $\mathbb{R}_\ell$.

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1If $X$ is an ordered set and $x \in X$, an immediate predecessor of $x$ is an element $y \in X$ such that $y < x$ and $(x, y) = \emptyset$. 1
$\mathbb{R}_\ell \times \mathbb{R}_\ell$. In each case it is a familiar topology. (Hint: Your answer may depend on the slope of $L$.)

**Problem 9:** Let $I = [0, 1]$ and compare the following three topologies on $I^2 = I \times I$.

- $T_1$ = the product topology on $I \times I$.
- $T_2$ = the dictionary order topology on $I \times I$.
- $T_3$ = the subspace topology on $I \times I$ inherited from the dictionary order topology on $\mathbb{R} \times \mathbb{R}$.  
