Problem 1: Let $k$ be a field and suppose that $A$ and $B$ are $k$-algebras. It can be shown that the $k$-vector space $A \otimes B$ has the structure of a $k$-algebra via the (well defined) multiplication

$$(a \otimes b)(a' \otimes b') := aa' \otimes bb',$$

for $a, a' \in A$ and $b, b' \in B$. Prove that we have the following isomorphism of $k$-algebras:

$k[x] \otimes k[y] \cong k[x, y]$.

Problem 2: Let $\phi^\lambda : \mathfrak{S}_n \to \mathbb{C}$ be the character of the $\mathfrak{S}_n$-module $M^\lambda$ for $\lambda \vdash n$. Find a formula for $\phi^\lambda_\lambda$, the value of $\phi^\lambda$ on the conjugacy class $K_\lambda \subseteq \mathfrak{S}_n$.

Problem 3: Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ be partitions of $n$. Characterize what it means for $\lambda$ to be covered by $\mu$ in

(1) lexicographic order, and
(2) dominance order.

Problem 4: Let $G$ be a finite group acting on a finite set $S$. The action of $G$ is called transitive if for all $s, t \in S$, there exists $g \in G$ such that $g.s = t$. The action of $G$ is called doubly transitive if for all $s_1, t_1, s_2, t_2 \in S$ with $s_1 \neq s_2$ and $t_1 \neq t_2$, there exists $g \in G$ such that $g.s_i = t_i$ for $i = 1, 2$. Prove the following.

(1) The orbits of $G$ partition $S$.
(2) The multiplicity of the trivial representation in $\mathbb{C}[S]$ is the number of orbits. Given $\lambda \vdash n$, what is the multiplicity of the trivial representation in $M^\lambda$?
(3) If $G$ acts doubly transitively on $S$ and $\chi : G \to \mathbb{C}$ is the character of $\mathbb{C}[S]$, prove that $\chi - 1$ is an irreducible character of $G$. (Hint: Fix $s \in S$ and use Frobenius reciprocity on the stabilizer $G_s \leq G$.)
(4) Conclude that the function $f : \mathfrak{S}_n \to \mathbb{C}$ given by

$$f(\pi) = (\text{number of fixed points of } \pi) - 1$$

is an irreducible character of $\mathfrak{S}_n$.

Problem 5: Consider the adjacent transpositions $\tau_k = (k, k+1) \in \mathfrak{S}_n$ for $k = 1, 2, \ldots, n-1$.

(1) Prove that $\{\tau_1, \tau_2, \ldots, \tau_{n-1}\}$ generate $\mathfrak{S}_n$ and satisfy the following Coxeter relations:

$$\tau_i^2 = e \quad \text{for } 1 \leq i \leq n-1$$
$$\tau_i \tau_j = \tau_j \tau_i \quad \text{for } |i - j| > 1$$
$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \quad \text{for } 1 \leq i \leq n-2$$

(2) Prove that if $G_n$ is the abstract group generated by the symbols $g_1, g_2, \ldots, g_{n-1}$ and subject to the relations above, then $G_n \cong \mathfrak{S}_n$. (Hint: Induct on $n$ using
the cosets of the subgroup generated by \( g_1, g_2, \ldots, g_{n-2} \).) This is a special case of Matsumoto’s Theorem in the theory of Coxeter groups.

**Problem 6:** Let \( \lambda \vdash n \). A standard \( \lambda \)-tableau is a bijective filling of the Ferrers diagram of \( \lambda \) with the numbers \( 1, 2, \ldots, n \) which is increasing across rows and down columns. An example of a standard \( \lambda \)-tableau \( T \) when \( \lambda = (4, 2, 2, 1) \vdash 9 \) is shown below.

\[
\begin{array}{cccc}
1 & 3 & 5 & 9 \\
2 & 4 &   &   \\
6 & 8 &   &   \\
7 &   &   &   \\
\end{array}
\]

Let \( SYT(\lambda) \) denote the set of all standard \( \lambda \)-tableaux.

1. A Dyck path of size \( n \) is a lattice path in \( \mathbb{Z}^2 \) starting at \((0, 0)\), ending at \((n, n)\), consisting only of north steps \( N = (0, 1) \) and east steps \( E = (1, 0) \), and staying weakly above the line \( y = x \). Give a simple bijection between \( SYT(n, n) \) and the set of all Dyck paths of size \( n \).

2. Young’s lattice is the partial order on the (infinite) set of all partitions given by \( \mu \leq \lambda \) if and only if \( \mu \subseteq \lambda \) (Ferrers diagram containment). Draw a reasonable portion of the bottom of Young’s lattice and prove that for any partition \( \lambda \), the maximal chains in the interval \([\emptyset, \lambda]\) in Young’s Lattice correspond bijectively to standard \( \lambda \)-tableaux. (Here \( \emptyset \) is the unique “empty” partition of 0.)

**Problem 7:** Let \( \lambda \vdash n \). A semistandard \( \lambda \)-tableau is a filling of the Ferrers diagram of \( \lambda \) with positive integers which is weakly increasing across rows and strictly increasing down columns. An example of a semistandard \( \lambda \)-tableau \( T \) when \( \lambda = (4, 2, 2, 1) \) is shown below.

\[
\begin{array}{cccc}
2 & 2 & 4 & 5 \\
3 & 4 &   &   \\
5 & 5 &   &   \\
7 &   &   &   \\
\end{array}
\]

The content of a semistandard tableau \( T \) is the sequence \( \mu = (\mu_1, \mu_2, \ldots) \), where \( \mu_i \) denotes the number of \( i \)'s in \( T \). For example, the content of the tableau shown above is \( \mu = (0, 2, 1, 2, 3, 0, 1, 0, 0, \ldots) \). Give combinatorial proofs of the following facts.

1. Suppose that \( \mu = (\mu_1, \mu_2, \ldots) \) and \( \mu' = (\mu'_1, \mu'_2, \ldots) \) are rearrangements of infinite sequences of nonnegative integers whose common sum is \( n \). Prove that the number of semistandard \( \lambda \)-tableaux with content \( \mu \) equals the number of semistandard \( \lambda \)-tableaux with content \( \mu' \). (Hint: It is enough to consider the case where \( \mu \) is obtained from \( \mu' \) by swapping \( \mu_i \) and \( \mu_{i+1} \) for some \( i \).)

2. Let \( \mu \vdash n \). The Kostka number \( K_{\lambda \mu} \) is the number of semistandard \( \lambda \)-tableaux with content \( \mu \). Prove that if \( K_{\lambda \mu} \neq 0 \), then \( \mu \preceq \lambda \) in dominance order. (The converse is also true, but you do not need to show this.)

**Problem 8:** Let \( G \) be a finite group and let \( H \leq G \) be an index two subgroup of \( G \). Prove the following.

1. \( H \) is normal in \( G \).
(2) Every conjugacy class of $G$ having nonempty intersection with $H$ is either becomes a conjugacy class of $H$ or splits into two conjugacy classes of $H$ of equal size. Furthermore, the conjugacy class $K$ of $G$ does not in $H$ if and only if some $k \in K$ commutes with some $g \notin H$.

(3) If $\chi : G \to \mathbb{C}$ is an irreducible character of $G$, then $\chi \downarrow_H : H \to \mathbb{C}$ is either irreducible or the sum of two inequivalent irreducible characters of $H$. Furthermore, $\chi \downarrow_H$ is irreducible if and only if $\chi(g) \neq 0$ for some $g \notin H$.

**Problem 9:** Find the character table of the alternating group $A_4$.

**Problem 10:** Given $X \subseteq S_n$, define group algebra elements $X^+, X^- \in \mathbb{C}[S_n]$ by

$$X^+ := \sum_{\pi \in X} \pi,$$

$$X^- := \sum_{\pi \in X} \text{sgn}(\pi)\pi.$$

For any group algebra element $a \in \mathbb{C}[S_n]$, the left ideal $\mathbb{C}[S_n]a$ generated by $a$ is a $\mathbb{C}$-vector space which carries an action of $S_n$ by left multiplication.

Let $\lambda \vdash n$ and let $T$ be any $\lambda$-tableau. The row-stabilizer $R_T \leq S_n$ is the subgroup of permutations which stabilize the entries in the rows of $T$. The column-stabilizer $C_T \leq S_n$ is the subgroup of permutations which stabilize the entries in the columns of $T$.

(1) Prove that $\mathbb{C}[S_n]R_T^+$ is isomorphic as an $S_n$-module to $M^\lambda$.

(2) Can you give a simple combinatorial description of the $S_n$-module $\mathbb{C}[S_n]C_T^-$ in terms of column tabloids?