FORMAT OF FINAL

- There will be 6 questions.
  - There should be a lot more time to focus on problems than on the midterms.
  - There will be a lot of multi-part questions on the final.
  - There will be a bonus survey question. The question will be: “Name one thing which you liked about the class and give one constructive criticism of the class.”

EXAM POLICY

- You may bring one 8.5 by 11 inch sheet of notes, which may be written on both sides, to each exam.
- You are to turn off your phones and other electronic devices during the exam.
  - No calculators will be allowed during exams.

SHOWING YOUR WORK AND PARTIAL CREDIT

The same policies as the midterms hold.

- You will be required to show a reasonable amount of work on each and every problem.
- You will of course be deducted points for calculation errors, but you will still get most of the possible points on a problem if your steps are, for the most part, correct.
• You are not required to do arithmetic to simplify your answer.
  – You do not have to evaluate logarithms (for example, you may leave \( \ln 2 \) as is) or powers of \( e \) (for example, you may leave \( e \) as is). Please do, however, evaluate trigonometric functions where possible (for example don’t leave \( \sin \pi \) as is, write 0 instead).
  * It may be helpful to brush up on the unit circle.

**WHAT’S GOING TO BE ON THE EXAM?**

The exam is cumulative. However, there will be more of an emphasis on the material we have covered since the last midterm (Sections 10.2-10.7, 9.1-9.2). Approximately half of the questions will be on the newer material.

**SELECTED TOPICS**

**WARNING:** This review does not cover everything that could be on the exam.

**FUNDAMENTAL THEOREM OF CALCULUS, PART II**

\[
\frac{d}{dx} \int_a^x f(t) \, dt = f(x)
\]

Further, \( F(x) = \int_a^x f(t) \, dt \) is the unique antiderivative of \( f(x) \) with the initial condition \( F(a) = 0 \).

**CHAIN RULE**

We can use the chain rule to cover the case where one or both of the limits of our integral are functions of \( x \).

\[
\frac{d}{dx} \int_{a(x)}^{b(x)} f(t) \, dt = f(b(x))b'(x) - f(a(x))a'(x)
\]

**EXAMPLE.** Evaluate \( \frac{d}{dx} \int_{\cos(x)}^{4x} e^{-t^2} \, dt \).
VOLUME PROBLEMS
Problems like those in Section 6.2 may appear on the test.

EXAMPLE. Find the volume of the cone with radius \( r \) and height \( h \).

VOLUMES OF REVOLUTION
Here’s the formula
\[
V = \pi \int_{a}^{b} R_{\text{outer}}^2 - R_{\text{inner}}^2 \, dx
\]

EXAMPLE. The region bounded by \( y = 8 - x^2 \) and \( y = x^2 \) is rotated about \( y = -2 \). Set up a definite integral representing the volume of the resulting solid.

INTEGRATION BY SUBSTITUTION AND BY PARTS
These are important to know. Here are some example problems.

EXAMPLE. Evaluate \( \int \sin^2 \theta \cos \theta \, d\theta \).

EXAMPLE. Evaluate \( \int (\ln(x))^2 \, dx \).

POLAR COORDINATES
Here are the conversion formulas between rectangular and polar coordinates:
\[
x = r \cos \theta \quad \text{and} \quad y = r \sin \theta.
\]

Another formula which is useful:
\[
r = \sqrt{x^2 + y^2} \quad \implies \quad r^2 = x^2 + y^2.
\]

Area bounded by a polar curve \( r = f(\theta) \):
\[
\text{area under the curve} = \frac{1}{2} \int_{a}^{b} f(\theta)^2 \, d\theta.
\]

Area between two polar curves \( r = f_1(\theta) \) and \( r = f_2(\theta) \):
\[
\text{area between curves} = \frac{1}{2} \int_{a}^{b} \left( f_2(\theta)^2 - f_1(\theta)^2 \right) \, d\theta.
\]
DOING POLAR AREA PROBLEMS

Here are some pointers on doing polar area problems:

• Try to sketch the graph of what you are trying to find the area of.
  – On some problems, the graph will be given to you
  – In general, a good strategy for drawing a graph is to plot a few points (you can make a table of \( \theta \) vs. \( r \) values).

• Identify what the curves bounding your region are.

• Figure out what your bounds are.
  – If your boundary is made of up multiple pieces, figure out to what \( \theta \) each endpoint corresponds. You can do this by calculating intersection points, or, if the endpoint is at the origin, by solving \( f(\theta) = 0 \).
  – Be careful of cases where \( r \) is negative. The \( \theta \) corresponding to a negative value of \( r \) as it appears on the graph is actually \( \theta + \pi \).
  – Your lower bound is always a smaller number than your upper-bound and the two bounds are within \( 2\pi \) of each other. The lower bound should be the extreme point of the curve in the clockwise direction. To adjust your bounds, add or subtract multiples of \( 2\pi \).

EXAMPLE. Find the region of overlap between the circles \( r = \sin \theta \) and \( r = \cos \theta \).

EULER'S THEOREM

\[ e^{i\theta} = \cos(\theta) + i\sin(\theta) \]

We can convert \( a + bi \) to \( re^{i\theta} \) by plotting the point \((a, b)\) and finding the angle and magnitude.

Other important formulas:

\[
\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin(i\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}
\]

EXAMPLE. Evaluate \( \int \sin(x) \cos(3x)\,dx \).
PARTIAL FRACTION EXPANSION

Know how to do it

EXAMPLE. Compute the partial fraction expansion of $\frac{1}{x^2(x^2 + 1)}$.

IMPROPER INTEGRALS

DOUBLY INFINITE INTEGRALS

$$\int_{-\infty}^{\infty} f(x) \, dx = \lim_{r \to -\infty} \int_{r}^{0} f(x) \, dx + \lim_{R \to \infty} \int_{0}^{R} f(x) \, dx$$

For one of these integrals to be convergent, both parts of the integral have to be convergent.

EXAMPLE. Evaluate $\int_{-\infty}^{\infty} \frac{x}{x^2 + 1} \, dx$ or show that it diverges.

APPROXIMATIONS

TRAPEZOIDAL APPROXIMATION

The $N^{th}$ trapezoidal approximation to $\int_{a}^{b} f(x) \, dx$ is

$$T_N = \frac{1}{2} \Delta x \left( f(a) + 2 f(a + \Delta x) + \cdots + 2 f(a + (N - 1)\Delta x) + f(b) \right)$$

where

$$\Delta x = \frac{b - a}{N}$$

MIDPOINT APPROXIMATION

The $N^{th}$ midpoint approximation to $\int_{a}^{b} f(x) \, dx$ is

$$M_N = \Delta x \left( f(a + [1/2] \Delta x) + f(a + [3/2] \Delta x) + \cdots + f(a + [N - 1/2] \Delta x) \right)$$

where

$$\Delta x = \frac{b - a}{N}$$

EXAMPLE. Calculate $M_4$ of $\int_{0}^{3} \sin(x) \, dx$. 5
COMPARISON THEOREM PROBLEMS
I also might give Comparison Theorem problems.

EXAMPLE. Determine if $\int_1^\infty \frac{dx}{x + x^2}$ converges.

EXAMPLE. Determine if $\int_1^\infty \frac{dx}{x - \sqrt{x}}$ converges.

SEQUENCES AND SERIES
We compute limits of sequences using standard limit laws.

TELESCOPING SERIES
We can compute the partial sums of a telescoping series to find its value.

EXAMPLE. Find $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$. Hint: $\frac{1}{n(n+2)} = \frac{1/2}{n} - \frac{1/2}{n+2}.$

GEOMETRIC SERIES

$$\sum_{n=0}^{\infty} cr^n = \frac{c}{1 - r}$$

EXAMPLE. Evaluate $\sum_{n=2}^{\infty} 7 \left( \frac{2}{5} \right)^n$.

ABSOLUTE CONVERGENCE
A series $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.

EXAMPLE. Does $\sum_{n=1}^{\infty} (-1)^n \sin \frac{n}{n^2}$ converge absolutely?

DIVERGENCE TEST
If $a_n$ does not converge to 0, then $\sum a_n$ does not converge either.

EXAMPLE. Does $\sum_{n=1}^{\infty} \frac{n!}{2^n}$ converge or diverge?
INTEGRAL TEST

Let \( a_n = f(n) \), where \( f(n) \) is positive, decreasing, and continuous for \( x \geq 1 \).

1. If \( \int_1^\infty f(x) \, dx \) converges, then \( \sum_{n=1}^\infty a_n \) converges.

2. If \( \int_1^\infty f(x) \, dx \) diverges, then \( \sum_{n=1}^\infty a_n \) diverges.

EXAMPLE. Does \( \sum_{n=2}^\infty \frac{1}{n \ln(n)} \) converge?

\( p \)-SERIES

The infinite series \( \sum_{n=1}^\infty \frac{1}{n^p} \) converges if \( p > 1 \) and diverges otherwise.

COMPARISON TEST

Assume that there exists a \( M > 0 \) such that \( 0 \leq a_n \leq b_n \) for \( n \geq M \)

1. If \( \sum_{n=1}^\infty b_n \) converges, so does \( \sum_{n=1}^\infty a_n \).

2. If \( \sum_{n=1}^\infty a_n \) diverges, so does \( \sum_{n=1}^\infty b_n \).

EXAMPLE. Does \( \sum_{n=1}^\infty \frac{1}{n - e^{-n}} \) converge or diverge?

LIMIT COMPARISON TEST

Let \( \{a_n\} \) and \( \{b_n\} \) be sequences and suppose

\[ \lim_{n \to \infty} \frac{a_n}{b_n} = L. \]

1. If \( L > 0 \), then \( \sum_{n=1}^\infty a_n \) converges if and only if \( \sum_{n=1}^\infty b_n \) converges.
2. If \( L = \infty \) and \( \sum_{n=1}^{\infty} a_n \) converges, then \( \sum_{n=1}^{\infty} b_n \) converges.

3. If \( L = 0 \) and \( \sum_{n=1}^{\infty} b_n \) converges, then \( \sum_{n=1}^{\infty} a_n \) converges.

**EXAMPLE.** Does \( \sum_{n=1}^{\infty} \frac{1}{n + e^{-n}} \) converge or diverge?

**RATIO TEST**

Assume that the following limit exists

\[
\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|
\]

1. If \( \rho < 1 \), then \( \sum a_n \) converges absolutely.
2. If \( \rho > 1 \), then \( \sum a_n \) diverges.
3. If \( \rho = 1 \), then the test is inconclusive (the series may converge or diverge).

**EXAMPLE.** Does \( \sum_{n=1}^{\infty} \frac{2^n}{n!} \) converge or diverge?

**ROOT TEST**

Assume that the following limit exists

\[
L = \lim_{n \to \infty} \sqrt[n]{|a_n|}
\]

1. If \( L < 1 \), then \( \sum a_n \) converges absolutely.
2. If \( L > 1 \), then \( \sum a_n \) diverges.
3. If \( L = 1 \), then the test is inconclusive (the series may converge or diverge).

**EXAMPLE.** Does \( \sum_{n=1}^{\infty} \left( \frac{\ln n}{n} \right)^n \) converge or diverge?
**POWER SERIES**

The **radius of convergence** of a power series comes by looking at the ratio of the terms. The radius of convergence of \( \sum_{n=1}^{\infty} a_n (x - b)^n \) is given by solving

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| |x - a| < 1
\]

**EXAMPLE.** Find the radius of convergence of \( \sum_{n=1}^{\infty} nx^n \).

**EXAMPLE.** Find the radius of convergence of \( \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} \).

**TAYLOR/MACLAURIN SERIES**

The **Taylor series** centered about \( x = a \) for a function \( f(x) \) is

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n
\]

The **Maclaurin series** is the Taylor series centered at \( x = 0 \).

**EXAMPLE.** Compute the Maclaurin series for \( f(x) = e^x \).

I am not going to focus as much on the brute force calculations of Taylor Series. Instead I want to focus on the most important known series.
There are shortcuts to building Maclaurin series from existing Maclaurin series.

**EXAMPLE.** Compute the Maclaurin series for \( x \cos(5x^2) \).

**EXAMPLE.** Find \( \frac{d^{2012}}{dx^{2012}} x \cos(5x^2) \).

*Hint:* Since we already have the series for \( x \cos(5x^2) \) and we know that the series has the equation:

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n
\]

we can get all the derivatives using the coefficients of our series.
APPROXIMATIONS

You can use the first $n$ terms of the Maclaurin series of a function to approximate it (as long as your point is within the radius of convergence).

DIFFERENTIAL EQUATIONS

The equation

$$y' = f(y)g(x)$$

is separable and

$$\frac{dy}{dx} = f(y)g(x) \quad \implies \quad \frac{dy}{f(y)} = g(x)dx$$

EXAMPLE. Solve $y' = (x - 1)y$.

With a separable equation, the solution always has a constant in the general solution.

EXAMPLE. Solve $y' = xy$ with the initial condition $y(0) = 1$. 