LOCALLY ALGEBRAIC VECTORS IN THE BREUIL-HERZIG CONSTRUCTION

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Abstract. For a fairly general reductive group $G_{/\mathbb{Q}_p}$, we explicitly compute the space of locally algebraic vectors in the Breuil-Herzig construction $\Pi(\rho)^{\text{ord}}$, for a potentially semistable Borel-valued representation $\rho$ of $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. The point being we deal with the whole representation, not just its socle – and we go beyond $\text{GL}_n(\mathbb{Q}_p)$. In the case of $\text{GL}_2(\mathbb{Q}_p)$, this relation is one of the key properties of the $p$-adic local Langlands correspondence. We give an application to $p$-adic local-global compatibility for $\Pi(\rho)^{\text{ord}}$ with no indecomposability assumptions.

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1. Introduction

The $p$-adic local Langlands correspondence for $\text{GL}_2(\mathbb{Q}_p)$ takes a continuous representation $\rho : \Gamma_{\mathbb{Q}_p} \to \text{GL}_2(E)$, defined over a finite extension $E/\mathbb{Q}_p$, and associates a Banach $E$-space $\Pi(\rho)$ with a unitary $\text{GL}_2(\mathbb{Q}_p)$-action. See [CDP] for the latest developments. One of its main features is its compatibility with the classical local Langlands correspondence, in the following sense. The locally algebraic vectors $\Pi(\rho)^{\text{alg}}$ are those which lie in an algebraic sub-representation of a compact open subgroup, and they may very well all be trivial. In fact, $\Pi(\rho)^{\text{alg}} \neq 0$ precisely when $\rho$ is potentially semistable with distinct Hodge-Tate weights – in which case $\Pi(\rho)^{\text{alg}}$
is given by the formula

\[ \Pi(\rho)^{\text{alg}} = \text{Sym}^{k-2}(E^2) \otimes_E \pi_{sm}(\rho), \]

where \( \pi_{sm}(\rho) \) corresponds to the Weil-Deligne representation \( WD(\rho) \), defined by Fontaine, via the local Langlands correspondence for \( \text{GL}_2(\mathbb{Q}_p) \) – or rather a "generic" extension thereof (a subtlety we can safely ignore for now). This compatibility is one of the main reasons why \( \Pi(-) \) plays such a fundamental role in Emerton and Kisin’s recent progress on the Fontaine-Mazur conjecture for two-dimensional representations of \( \Gamma = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \), cf. \[Eme\] for example.

For \( \text{GL}_n(\mathbb{Q}_p) \) one still doesn’t quite know how to define the \( p \)-adic local Langlands correspondence, although a globally defined candidate \( \Pi(-)^{\text{ord}} \) has recently been proposed (with very promising properties), see \[Pat\], which employs an intricate version of the Taylor-Wiles-Kisin patching argument at infinite level. When the representation \( \rho : \Gamma \rightarrow \text{GL}_n(E) \) is triangular, i.e. maps into the Borel \( \hat{B}(E) \), Breuil and Herzig give a purely local construction of a Banach representation \( \Pi(\rho)^{\text{ord}} \) of \( \text{GL}_n(\mathbb{Q}_p) \), which is expected to be the part of \( \Pi(\rho)^{\text{g}} \) which can be built out of principal series. In fact, the construction in \[BH14\] is more general, and deals with fairly general reductive \( p \)-adic groups \( G(\mathbb{Q}_p) \). The purpose of this note is to prove the analogue of 1.1 for the Breuil-Herzig construction \( \Pi(-)^{\text{ord}} \), for any \( G \) satisfying the usual weak assumption that \( Z_G \) and \( Z_G^{\text{der}} \) are connected.

We now present the main result of this paper in more detail. We begin by briefly introducing the notation and terminology in use throughout.

Throughout, \( G_{/\mathbb{Q}_p} \) is a split connected reductive group, with a choice of Borel pair \( B \supset T \). As in \[BH14\], we always assume that \( Z_G \) is connected and \( G^{\text{der}} \) is simply connected (assumption 2.1 below). Once and for all, we pick a "twisting element" \( \theta \in X(T) \). That is, a character such that \( \langle \theta, \alpha^\vee \rangle = 1 \) for all simple roots \( \alpha \) (for example, take \( \theta \) to be the sum of the fundamental weights). The terminology is taken from \[BG14\] (their Definition 5.2.1, p. 27). We consider continuous \( \hat{B}(E) \)-valued representations,

\[ \rho : \Gamma_{\mathbb{Q}_p} = \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \rightarrow \hat{B}(E) \subset \hat{G}(E), \]

where \( E/\mathbb{Q}_p \) is a finite extension. To avoid degeneracies, we always assume \( \rho \) is good – which means the Zariski closure of \( \rho(\Gamma_{\mathbb{Q}_p}) \) is as small as possible among all \( \hat{B}(E) \)-conjugates (see Definition 3.2.4 in \[BH14\], or Section 2.3 for details). Following \[BH14\], we consider the semi-simplification \( \hat{\chi}_{\rho} : \Gamma_{\mathbb{Q}_p} \rightarrow \hat{T}(E) \), and the character \( \chi_{\rho} : T(\mathbb{Q}_p) \rightarrow E^\times \) corresponding to \( \hat{\chi}_{\rho} \) via class field theory (see 2.2 below). We say \( \rho \) is generic if \( \alpha^\vee \circ \hat{\chi}_{\rho} \notin \{1, \epsilon^{\pm 1}\} \) for all roots \( \alpha \) (where \( \epsilon \) is the \( p \)-adic cyclotomic character). To a good generic \( \rho \), the Breuil-Herzig construction attaches
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a unitary continuous representation $\Pi(\rho)^{ord}$ of $G(\mathbb{Q}_p)$ on an $E$-Banach space – which is expected to account for the "ordinary" part of the (yet to be defined) $p$-adic local Langlands correspondence $\Pi(\rho)^?$. That is, $\Pi(\rho)^{ord}$ should be the maximal closed invariant subspace of $\Pi(\rho)^?$ whose constituents are (subquotients of) continuous principal series. The construction of $\Pi(\rho)^{ord}$ is based on the belief that there should be an analogue $F$ of Colmez’s Montreal functor such that $L^\otimes \circ \rho = F(\Pi(\rho)^?)$, where $L^\otimes$ is the tensor product of all the fundamental algebraic representations of $\hat{G}$. In a recent preprint [Br14], Breuil gives strong evidence for this: In Corollary 9.8 of loc. cit. he defines an exact contravariant functor $F$ from representations built from principal series to etale $(\phi, \Gamma)$-modules, which enjoys all the desired properties. In fact, his construction is more general, and involves projective limits of etale $(\phi, \Gamma)$-modules.

As in the GL(2)-case, $\Pi(\rho)^{ord}$ should admit locally algebraic vectors precisely when $\rho$ is potentially semistable\footnote{Recall that, by definition, $\rho : \Gamma_{\mathbb{Q}_p} \to \hat{G}(E)$ is potentially semistable if $r \circ \rho$ is potentially semistable for every algebraic representation $r : \hat{G} \to \text{GL}(W)$. See Definition 1.1.1 in [Win].} and regular. For a general group $G$, and a general $\rho$, our definition of regular is a bit ad hoc (3.1). We take it to mean $\hat{G}_\rho = \rho(I_{\mathbb{Q}_p})$ has a faithful representation with one-dimensional weight spaces (for $G = \text{GL}(n)$, and $\rho$ de Rham, this means the Hodge-Tate weights are distinct). Regularity is needed to draw the conclusion that $\chi_\rho$ locally algebraic $\Rightarrow$ $\rho$ potentially semistable (the converse being relatively clear). This implication reduces easily to the GL($n$)-case, which can be found in [Ne93], as Theorem 1.30. When $\chi_\rho = \chi_\rho,\text{alg} \cdot \chi_\rho,\text{sm}$ is locally algebraic, $\chi_\rho,\text{alg}^+ \in X_+(T)$ denotes the dominant Weyl-conjugate of the algebraic part $\chi_\rho,\text{alg}$. Our main result computes $\Pi(\rho)^{ord,\text{alg}}$ in terms of the classical local Langlands correspondence, a la the Breuil-Schneider recipe in the GL($n$)-case, introduced in [BrSc] (see also its predecessor [ScTe]).

**Theorem 1.2.** Suppose $\rho : \Gamma_{\mathbb{Q}_p} \to \hat{B}(E)$ is good, residually\footnote{This can be slightly weakened. Under assumption 2.3, generic is enough.} generic, and regular\footnote{Strictly speaking, regularity is only needed in part (1) of the Theorem.}. Let $V$ be an irreducible algebraic representation of $G(\mathbb{Q}_p)$, defined over $E$, of highest weight $\lambda \in X_+(T)$ relative to $B$. Then the following holds.

1. If $\Pi(\rho)^{ord,V-\text{alg}} \neq 0$, then $\rho$ is potentially semistable, and $\chi_\rho,\text{alg}^+ = \lambda + \theta$.
2. If $\rho$ is potentially semistable, with $\chi_\rho,\text{alg} = \lambda + \theta$, then

$$\Pi(\rho)^{ord,\text{alg}} = \Pi(\rho)^{ord,V-\text{alg}} = V \otimes_E (\text{Ind}_{B^-_{\mathbb{Q}_p}}^{G(\mathbb{Q}_p)} \chi_{\rho,\text{sm}} \cdot |\theta|^{-1})^{C^\infty}. $$

(The superscript $C^\infty$ means we take the smooth parabolic induction, unnormalized.)

The GL($n$)-case of this Theorem is undoubtedly known to the experts. In fact, Theorem 8.9 (together with Lemma 8.2) in [Br13] computes the locally algebraic
vectors in the socle \( \text{soc}\text{GL}_n(\mathbb{Q}_p)\Pi(\rho)^{\text{ord}} \). Our goal is twofold: (a) To show there are no locally algebraic vectors in the cosocle, and (b) to extend this to an arbitrary group \( G \) as above – which requires a bit of delicate Lie-theoretic book-keeping. In some sense, the heart of the argument is Proposition 4.1 below, which shows that the Weyl element \( w_\rho \) such that \( w_\rho^{-1}(\chi_{\rho,\text{alg}}) \) is dominant, conjugates the image of \( \rho \) into \( \hat{B} \). The latter is a \( G \)-version of the fact from \( p \)-adic Hodge theory that there are no non-split extensions \( 1 \rightarrow \epsilon \rightarrow h \rightarrow 0 \). (A good reference for this is the discussion on p. 122 in [BC09]. See also Example 3.9 in [BK].) Proposition 4.1 is what allows us to control the constituents of the cosocle in Section 5.

In the last Chapter we give a quick application of Proposition 4.1 to the \( p \)-adic local-global compatibility conjecture in [BH14]. Their Theorem 4.4.8 shows that \( \Pi(-)^{\text{ord}} \) of a modular representation \( r \) injects into a space of \( p \)-adic modular forms, assuming that \( \overline{r} \) is totally indecomposable at the places above \( p \). We prove a variant in which we allow \( \overline{r} \) to be arbitrarily decomposable, but only get a nonzero map.

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2. Review of the Breuil-Herzig construction

2.1. Group-theoretic notation. We adopt the notation and setup from [BH14], which we briefly recall. We fix a split connected reductive group \( G_{/\mathbb{Z}_p} \), endowed with a choice of split maximal torus \( T \), and a Borel subgroup \( B \supset T \), whose opposite is denoted \( B^- \). Let \( (X(T), R, X^\vee(T), R^\vee) \) be the corresponding root datum, and let \( S \subset R^+ \) be the simple roots associated with \( B \). For each \( \alpha \in S \), we have the simple reflection \( s_\alpha \in \text{Aut}X(T) \) given by \( s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha \). They generate the Weyl group \( W \). The dual based root datum determines (up to inner automorphisms) a dual triple \( (\hat{G}, \hat{B}, \hat{T}) \), which we view as being defined (and split) over \( \mathcal{O}_E \), for a fixed finite extension \( E/\mathbb{Q}_p \). Similarly, we have simple reflections \( s_\alpha^\vee \in \text{Aut}X(\hat{T}) \), for \( \alpha \in S \), which generate the Weyl group \( \hat{W} \simeq W \) (via the duality, \( s_\alpha^\vee \leftrightarrow s_\alpha \)).

Assumption 2.1. The center \( Z_G \) is connected, and the derived group \( G^{\text{der}} \) is simply connected (equivalently, \( Z_G \) is connected – see Proposition 2.1.1 in [BH14]).

Thus, both \( G \) and \( \hat{G} \) admit fundamental weights, \( \{\lambda_\alpha\}_{\alpha \in S} \) and \( \{\lambda_\alpha^\vee\}_{\alpha \in S} \) respectively. For example, \( \lambda_\alpha \in X(T) \) is a (necessarily dominant) weight such that \( \langle \lambda_\alpha, \beta^\vee \rangle = \delta_{\alpha,\beta} \) for \( \beta \in S \). This determines \( \lambda_\alpha \) modulo \( X^0(T) \simeq X(G) \). Once and for all, we choose a twisting element \( \theta \) (in the sense of [BG14]). That is, a \( \theta \in X(T) \) with \( \langle \theta, \alpha^\vee \rangle = 1 \) for all \( \alpha \in S \). For example, \( \theta = \sum_{\alpha \in S} \lambda_\alpha - \text{everything defined modulo } X^0(T) \).
Let $U \subset B$ be the unipotent radical. For each $\alpha \in R$, we have an associated root subgroup $U_\alpha \simeq \mathbb{G}_\alpha$. Moreover, $U_\alpha \subset B$ precisely when $\alpha \in R^+$. A subset $C \subset R$ is closed if $\forall \alpha, \beta \in C$, $\alpha + \beta \in C$. Thus $C$ is the set of roots of $U_C$, the group generated by $\{U_\alpha\}_{\alpha \in C}$. We let $B_C = TU_C$. With $C$, we associate the following subset of the Weyl group (see Lemma 2.3.6 in [BH14]),

$$W_C = \{ w \in W : w^{-1}(C) \subset R^+ \} = \{ w \in W : \hat{w}^{-1}B_C\hat{w} \subset B \}.$$  

(Here $\hat{w} \in N_G(T)$ is any representative of $w$.) Analogous notation and terminology will be used for the dual group $\hat{G}$ below.

2.2. Galois-theoretic notation. Our starting point is a continuous homomorphism ("representation") taking values in $\hat{B}$,

$$\rho : \Gamma_{\mathbb{Q}_p} = \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \longrightarrow \hat{G}(E), \quad \rho(\Gamma_{\mathbb{Q}_p}) \subset \hat{B}(E).$$

These are called ordinary in [BH14], but this can be somewhat misleading since this usually pertains to semistable representations. Pink has a notion of quasi-ordinary [Pi98], which seems more appropriate. We prefer to just call $\rho$ triangular. Regardless, we let $C_\rho \subset R^{\vee}$ be the smallest closed subset such that $\rho(\Gamma_{\mathbb{Q}_p}) \subset \hat{B}_{C_\rho}(E)$. By composition with the projection $\hat{B} \to \hat{T}$, we obtain the "semisimplification",

$$\hat{\chi}_\rho : \Gamma_{\mathbb{Q}_p} \longrightarrow \hat{B}(E) \to \hat{T}(E).$$

This is continuous, and therefore takes values in the maximal compact subgroup $\hat{T}(O_E)$. Via local class field theory, we attach a continuous character $\chi_\rho : T(\mathbb{Q}_p) \to O_E^\times$ as follows: First recall that, for any $\mathbb{Z}_p$-algebra $A$,

$$T(A) = \text{Hom}(X(T), A^\times) = \text{Hom}(X(T), \mathbb{Z}) \otimes_{\mathbb{Z}} A^\times = X(\hat{T}) \otimes_{\mathbb{Z}} A^\times.$$  

Taking $A = \mathbb{Q}_p$, and composing with the Artin map $\mathbb{Q}_p^\times \hookrightarrow \Gamma_{\mathbb{Q}_p}^{ab}$, gives rise to $\chi_\rho$,

$$T(\mathbb{Q}_p) = X(\hat{T}) \otimes_{\mathbb{Z}} \mathbb{Q}_p^\times \hookrightarrow X(\hat{T}) \otimes_{\mathbb{Z}} \Gamma_{\mathbb{Q}_p}^{ab} \xrightarrow{1 \otimes \hat{\chi}_\rho} X(\hat{T}) \otimes_{\mathbb{Z}} \hat{T}(O_E) \longrightarrow O_E^\times,$$

where the last map is just evaluation. By construction, $\chi_\rho$ is continuous.

2.3. The construction. We say $\rho$ is good if $C_\rho \subset C_{\mathbb{Q}_p^{b_{\text{ord}}}}$, for all $b \in \hat{B}(E)$. By 3.2.3 in [BH14], our representation $\rho$ admits a good $\hat{B}(E)$-conjugate if $\alpha^{\vee} \circ \hat{\chi}_\rho \neq 1$, for all $\alpha \in R^+$. We will always assume something stronger. Namely, that $\rho$ is generic (3.3.1 in loc. cit.), which means $\alpha^{\vee} \circ \hat{\chi}_\rho \notin \{ 1, \epsilon^{\pm 1} \}$, for all $\alpha \in R$. (Here $\epsilon$ is the $p$-adic cyclotomic character.) Thus, from now on, $\rho$ is good and generic. For such $\rho$, Breuil and Herzig define a corresponding $E$-Banach space representation $\Pi(\rho)^{\text{ord}}$ of $G(\mathbb{Q}_p)$, which is expected to be the largest sub-representation of a conjectural $p$-adic local Langlands correspondence $\Pi(\rho)$ built out of continuous principal series.
The definition of $\Pi(\rho)^{ord}$ is based on the belief\footnote{Now strongly supported by the construction in [Br14]!} that there should be a ”Colmez” functor $F$ such that

$$F(\Pi(\rho)) \cong L^\otimes \circ \rho, \quad L^\otimes = \otimes_{\alpha \in S} L(\lambda_\alpha^\vee).$$

The submodule structure of $\Pi(\rho)^{ord}$ should therefore reflect that of the ”ordinary” part $(L^\otimes_{BC_p})^{ord}$, worked out in Theorem 2.4.1 of [BH14]. Thus, it decomposes as

$$\Pi(\rho)^{ord} = \oplus_{w \in W_c} \Pi(\rho)_{C_p, w},$$

where $\Pi(\rho)_{C_p, w}$ is a certain direct limit $\varinjlim_I \Pi(\rho)_I$, with $I \subset w(S^\vee) \cap C_p$ ranging over (the finitely many) subsets of pairwise orthogonal roots. Here $\Pi(\rho)_I$ is a parabolically induced representation,

$$\Pi(\rho)_I = (\text{Ind}_{P_{\rho, I}(Q_p)}^{G(Q_p)} \tilde{\Pi}(\rho)_{I})^{c_0},$$

induced from the parabolic $P_J \supset B^-$ with Levi $G_J \simeq T_J' \times GL_2^J$ (see 3.1.4 in loc. cit.). Here $J$ is the corresponding set of simple roots, $J = w^{-1}(I)^\vee \subset S$. Finally, $\tilde{\Pi}(\rho)_I$ is a representation of $G_J(Q_p)$, characterized by its socle filtration (3.3.3 in loc. cit.) – essentially it’s built out of the $p$-adic local Langlands correspondence for the copies of $GL_2(Q_p)$ sitting in $G_J(Q_p)$ (a copy for each simple root $\beta \in J$).

We will not need the details of the construction in this paper. It is expected that the summand $\Pi(\rho)_{C_p, w}$ can be characterized by its socle filtration. More precisely, one expects the following, which is Conjecture 3.5.1, p. 40, in [BH14].

**Conjecture 2.2.** There is a unique admissible unitary continuous representation $\Pi(\rho)_{C_p, w}$ of $G(Q_p)$ over $E$ with socle filtration $Fil_j \Pi(\rho)_{C_p, w}$ such that $\forall j \geq 0$,

$$\text{Gr}_j \Pi(\rho)_{C_p, w} = Fil_j \Pi(\rho)_{C_p, w} / Fil_{j-1} \Pi(\rho)_{C_p, w} \simeq \bigoplus_{I \subset w(S^\vee) \cap C_p, |I| = j} (\text{Ind}_{B(J)}^{G(Q_p)}(\prod_{\alpha \in I^\vee} s_{\alpha})w)^{-1}(\chi_\rho) \cdot (\epsilon^{-1} \circ \theta))^{c_0}.$$  

Here $I \subset w(S^\vee) \cap C_p$ runs over subsets of pairwise orthogonal roots, $|I| = j$.

Breuil and Herzog also speculate that $\Pi(\rho)_{C_p, w}$ should be characterized by the following weaker conditions:

- $\text{soc}_{G(Q_p)} \Pi(\rho)_{C_p, w} = \text{Fil}_0 \Pi(\rho)_{C_p, w} \simeq (\text{Ind}_{B(J)}^{G(Q_p)}(\prod_{\alpha \in I^\vee} s_{\alpha})w)^{-1}(\chi_\rho) \cdot (\epsilon^{-1} \circ \theta))^{c_0}.$
- All constituents of $\Pi(\rho)_{C_p, w}$ occur with multiplicity one, and have the form

$$(\text{Ind}_{B(J)}^{G(Q_p)}(\prod_{\alpha \in I^\vee} s_{\alpha})w)^{-1}(\chi_\rho) \cdot (\epsilon^{-1} \circ \theta))^{c_0}$$

for varying subsets $I \subset w(S^\vee) \cap C_p$ of pairwise orthogonal roots.

As mentioned on p. 40 in [BH14], this is known if $\tilde{\chi}_\rho \circ \alpha^\vee \neq \omega$ for all $\alpha \in w(S)$, with $w \in W_\rho$ (a subset of the Weyl group comprising all $(\prod_{\alpha \in I^\vee} s_{\alpha})w$, see (14) on p.
30 in loc. cit., it contains $W_{C_p}$ (but it may be bigger). Here $\omega = \bar{\epsilon}$ denotes the mod $p$ cyclotomic character, viewed as a character $Q_p^\times \to k_E^\times$. Indeed, if this condition holds, all the continuous principal series above are topologically irreducible – due to independent works of Ollivier ($G = GL_n$) and Abe ($G$ split), see [Oll], Theorem 4, and [Abe], Theorem 1.3, respectively. We will impose this "weak genericity" condition on $\bar{\rho}$ in what follows.

Assumption 2.3. $\bar{\chi}_\rho \circ \alpha^\vee \neq \omega$ for all $\alpha \in w(S)$, with $w \in W_{C_p}$.

Remark 2.4. In a recent preprint, J. Hauseux proves Breuil-Herzig’s Conjecture 2.2 under the assumption that the principal series involved are topologically irreducible (which is slightly weaker than 2.3). See Theorem 4.1.4 and its Corollary 4.1.6 in [Hau]. This irreducibility assumption should be vacuous (according to the “folklore” Conjecture 3.1.2 in [BH14]. See also [Sch], 2.5).

3. Regular and ordinary implies semistable

A result of Nekovar (and independently Perrin-Riou) says that "ordinary implies semistable". See Propositions 1.24–1.28 in [Ne93], nicely summarized as Lemma 3.1.4 in [GG12]. Here we present a variant of this result for $\hat{G}$-valued triangular representations $\rho : \Gamma_{Q_p} \to \hat{B}(E)$, as above, for which $\hat{\chi}_\rho$ is potentially crystalline. We need to impose a certain regularity condition.

Definition 3.1. Let $\hat{G}_\rho = \overline{\rho(I_{Q_p})^\circ}$ be the neutral component of the Zariski closure of the image of inertia (a connected solvable subgroup of $\hat{B}$). Let $\hat{T}_\rho \subset \hat{T}$ be a maximal torus. We say $\rho$ is regular if the group $\hat{G}_\rho$ admits a faithful representation $\hat{G}_\rho \hookrightarrow \text{GL}(V)$ with one-dimensional weight spaces for $\hat{T}_\rho$. That is, one has a decomposition $V = \oplus_{\mu \in X(\hat{T}_\rho)} V(\mu)$, where $\text{dim} V(\mu) \leq 1$ for all $\mu$.

Thus, a non-trivial $\rho$ for which $\rho(I_{Q_p})$ is unipotent is never regular. On the other hand, $\rho$ is always regular if $\rho(I_{Q_p})$ consists of semisimple elements (that is, when $\rho|_{I_{Q_p}}$ is "diagonalizable") – just take $V$ to be the direct sum of a basis for $X(\hat{T}_\rho)$.

Example 3.2. Suppose $\hat{G} = \text{GL}(n)$. Let $\rho$ be a representation which satisfies the condition $(\alpha^\vee \circ \hat{\chi}_\rho)|_J \neq 1$, for all $\alpha \in R$, and all open subgroups $J \subset I_{Q_p}$. (Note that if $\rho$ happens to be potentially semistable, this simply says the Hodge-Tate weights are distinct.) Such a $\rho$ is regular, in the sense of 3.1. Indeed, take $V$ to be the standard $n$-dimensional representation, restricted to $\hat{G}_\rho$. Its weights are $\{\mu_i\}$, where $\mu_i$ picks the $i$th diagonal entry. We must verify that all the restrictions $\mu_i|_{\hat{T}_\rho}$ are distinct. Suppose $\mu_i - \mu_j$ becomes trivial on $\hat{T}_\rho$. This difference is one of the roots, $\mu_i - \mu_j = \alpha^\vee$, for some $\alpha \in R$. The next lemma identifies $\hat{T}_\rho$ with $\overline{\chi}_\rho(I_{Q_p})$, from which we deduce that $\alpha^\vee \circ \hat{\chi}_\rho$ is trivial on small enough $J$. Contradiction.
A similar argument works for $\hat{G} = \text{GSp}(2n)$, using the description of the roots in Example 2.1.3 of [BH14], say.

We return to the general setup.

**Lemma 3.3.** $\hat{T}_\rho \xrightarrow{\sim} \hat{G}^\text{red}_\rho \xrightarrow{\sim} \hat{G}_{\hat{\chi}_\rho}$.

**Proof.** The first map is the inclusion $\hat{T}_\rho \subset \hat{G}_\rho$ composed with the natural projection $\hat{G}_\rho \to \hat{G}_\rho^\text{red}$ onto the maximal reductive quotient. It’s an isomorphism, as follows from the decomposition $\hat{G}_\rho = \hat{T}_\rho \times \hat{G}_{\rho,u}$ (using that $\hat{G}_\rho$ is connected solvable).

The second map is obtained as follows: The projection $\hat{B} \to \hat{T}$ maps $\rho(I_{\mathbb{Q}_p})$ into $\hat{\chi}_\rho(I_{\mathbb{Q}_p})$, and therefore restricts to a map $\hat{G}_\rho \to \hat{G}_{\hat{\chi}_\rho}$. Its image is closed (a general fact about homomorphisms between linear algebraic groups), and contains $\hat{\chi}_\rho(J)$, for small enough $J$. It’s therefore onto. Moreover, the kernel $\hat{G}_\rho \cap \hat{U}$ is the unipotent radical $\hat{G}_{\rho,u}$, as is easily verified. 

This leads to the main result of this section, which generalizes Nekovar’s result in the $\text{GL}(n)$-case.

**Proposition 3.4.** Assume $\rho : \Gamma_{\mathbb{Q}_p} \to \hat{B}(E)$ is regular, as defined in 3.1. Then $\rho$ is potentially semistable if and only if $\hat{\chi}_\rho$ is potentially crystalline. (Equivalently, $\chi_\rho$ is locally algebraic).

**Proof.** The parenthesis is immediate: Via class field theory, potentially crystalline characters $\hat{\chi} : \Gamma_{\mathbb{Q}_p} \to E^\times$ correspond to locally algebraic characters $\chi : \mathbb{Q}_p^\times \to E^\times$. Also, the ”only if” is almost trivial: Pick a faithful representation $r : \hat{T} \hookrightarrow \text{GL}(W)$. We must show $r \circ \hat{\chi}_\rho$ is potentially crystalline. It’s the composition

$$\Gamma_{\mathbb{Q}_p} \xrightarrow{\rho} \hat{B} \to \hat{T} \hookrightarrow \text{GL}(W),$$

which is potentially semistable, since $\rho$ is. Finally, $N = 0$ since $\hat{\chi}_\rho$ is a character.

For the converse, regularity plays a role. Assuming $\hat{\chi}_\rho$ is potentially crystalline, we will explain why $\rho_V : J \to \text{GL}(V)$ is potentially semistable, where $V$ is the representation from 3.1, and $\rho_V$ is the composition $J \to \hat{G}_\rho \hookrightarrow \text{GL}(V)$. Here $J \subset I_{\mathbb{Q}_p}$ is an open subgroup such that $\rho(J) \subset \hat{G}_\rho$. (Connectedness is the issue here. We may take $J = I_{\mathbb{Q}_p}$ if the Zariski closure of $\rho(I_{\mathbb{Q}_p})$ is connected.) We do this by showing $\rho_V$ is ordinary (in the sense of Definition 3.1.3 of [GG12]), of regular weight, and then referring to [Ne93].

Choose a nonzero vector $e_\mu \in V(\mu)$ in each weight space. After suitably ordering the weights, say $\{\mu_1, \ldots, \mu_n\}$, the basis $(e_\mu)$ identifies $\text{GL}(V) \simeq \text{GL}(n)$ in such a way that the pair $(\hat{G}_\rho, \hat{T}_\rho)$ maps into the upper-triangular pair $(\hat{B}_{\text{GL}(n)}, \hat{T}_{\text{GL}(n)})$. 
Thus, for certain characters $\chi_i : J \to \bar{\mathbb{Q}}_p^\times$, the representation $\rho_V$ has the form

$$\rho_V = \begin{pmatrix} \chi_1 & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & \chi_n \end{pmatrix}.$$ 

We claim that $\chi_i = \mu_i \circ (\hat{\chi}_\rho|_J)$, for all $i$, where we view $\hat{\chi}_\rho|_J$ as a character $J \to \hat{T}_\rho$ via the isomorphisms in 3.3. This is a straightforward unwinding of definitions: The map $\hat{G}_\rho \hookrightarrow \hat{B}_{GL(n)}$ restricts to the map $\hat{T}_\rho \hookrightarrow \hat{T}_{GL(n)}$ given by $\mu_1 \oplus \cdots \oplus \mu_n$. Now, since $\hat{\chi}_\rho$ is potentially crystalline, so are the $\chi_i$. Moreover, $\hat{\chi}_\rho(J)$ is dense in $\hat{T}_\rho$, since $J$ has finite index in $I_{\mathbb{Q}_p}$. Therefore the $\chi_i$ are distinct, since $\mu_i \neq \mu_j$ for $i \neq j$. By choosing $J$ even smaller, we may assume that $\chi_i = (\epsilon|_J)^{h_i}$, for distinct integers $h_i$. Referring to Definition 3.1.3 in [GG12], we see that $\rho_V$ is ordinary, of some weight $\lambda$, if $h_1 > \cdots > h_n$. This can be arranged by taking a suitable conjugate of $\rho_V$: Look at neighbor diagonal entries. Suppose $h_i \leq h_{i+1}$. Then in fact $h_i < h_{i+1}$, since we know the weights are distinct. As is well-known from $p$-adic Hodge theory, the extension $\begin{pmatrix} \chi_i \\ \chi_{i+1} \end{pmatrix}$ must therefore split. Thus, taking a suitable conjugate of $\rho_V$, we can turn this $2 \times 2$ block into $\begin{pmatrix} \chi_{i+1} \\ \chi_i \end{pmatrix}$, while leaving $\rho_V$ triangular. After a sequence of these adjacent swaps, we may assume the $h_i$ decrease down the diagonal – that is, $\rho_V$ is ordinary. By part (1) of Lemma 3.1.4 in [GG12] (due to Nekovar) we conclude that $\rho_V$ is potentially semistable. □

**Remark 3.5.** We point out that regularity is needed in 3.4; extensions of de Rham representations need not be de Rham. At the bottom of p. 9 in [Ber] is given an example of a two-dimensional representation $\rho = \begin{pmatrix} 1 & * \\ 1 & 1 \end{pmatrix}$ which is not even Hodge-Tate (the extension is essentially given by $\log_p$).

4. **The dominant Weyl-conjugate**

In this section we elaborate on the argument at the end of the proof of 3.4. Thus, let $\rho : \Gamma_{\mathbb{Q}_p} \to \hat{B}(E)$ be a triangular representation, and assume $\hat{\chi}_\rho$ is potentially crystalline. In other words, $\chi_\rho$ is locally algebraic. Factor it into an algebraic part and a smooth part, $\chi_\rho = \chi_{\rho,alg} \cdot \chi_{\rho,sm}$. Here $\chi_{\rho,alg} \in X(T)$ need not be dominant, but as long as we assume it lies away from the walls, it has a unique dominant Weyl-conjugate which we denote $\chi_{\rho,alg}^+ \in X_+(T)$. Since $W$ acts simply transitive on the Weyl chambers, there is a unique $w_\rho \in W$ such that $\chi_{\rho,alg}^+ = w_\rho^{-1}(\chi_{\rho,alg})$.

**Proposition 4.1.** Suppose the representation $\rho : \Gamma_{\mathbb{Q}_p} \to \hat{B}(E)$ is good, generic, and potentially semistable. Then $w_\rho \in W_{C_\rho}$ (viewed as a subset of $W \simeq \hat{W}$).
Remark 4.2. In the GL(2)-case, this is the fact used in 3.4 that $\left( e^{h_1} \ast e^{h_2} \right)$ must split if $h_1 < h_2$. We will reduce the general case (for any $G$) to this special case, by suitably adapting the argument at the very end of the proof of 3.4 – which deals with the GL(n)-case. Proposition 4.1 will play a crucial role in the proof of the main result, in the next section.

Proof. We proceed by induction on the length $\ell(w_\rho)$. If the length is zero, $w_\rho = 1$, and we’re done since $C_\rho \subset R^+ \lor$. Assume the Proposition holds for all $\rho'$ (satisfying the given hypotheses) of length $\ell(w_\rho') < \ell(w_\rho)$. To show that $w_\rho \in W_{C_\rho}$, once and for all we pick a simple root $\beta \in S$ such that $\langle \chi_\rho, \text{alg}, \beta \lor \rangle < 0$. (Such a $\beta$ exists since $\ell(w_\rho) > 0$. That is, $\chi_\rho, \text{alg}$ is not dominant.) View $\beta \lor$ as an element of $S \lor \subset X(\hat{T})$, and look at the simple reflection $s_\beta \in \hat{W}$, which corresponds to $s_\beta \in W$ under the identification of Weyl groups $W \simeq \hat{W}$. Pick a representative $s := s_\beta \in N_{G}(\hat{T})$, and introduce the conjugate $\rho' := s \rho s^{-1}$. A priori this is just a continuous homomorphism $\rho' : \Gamma_{Q_p} \to \hat{G}(E)$. We claim it satisfies all the criteria in 4.1, plus the induction hypothesis.

Step 1. $C_\rho$ does not contain $\beta \lor$.

To see this, let $\hat{P} \supseteq \hat{B}$ be the standard parabolic associated with $\{\beta \lor\}$. It is generated by $\hat{B}$ and $s$, and has a Levi decomposition $\hat{P} = \hat{M} \hat{N}$, where $\hat{M} \simeq \hat{G}_{\beta \lor}$ in the notation of [BH14] (see p. 24). By their Lemma 3.1.4, there is an isomorphism $\hat{M} \simeq \hat{T}' \times \text{GL}(2)$ for a subtorus $\hat{T}' \subset \hat{T}$, central in $\hat{M}$. (This uses Assumption 2.1.) We can arrange that, under this isomorphism, the Borel pair $(\hat{B} \cap \hat{M}, \hat{T})$ corresponds to the upper-triangular Borel pair in GL(2) – times $\hat{T}'$. In particular, the unique simple root $\beta \lor$ of $\hat{M}$ corresponds to $1 \otimes \alpha_{\text{GL}(2)}$, where $\alpha_{\text{GL}(2)}$ denotes the simple root of GL(2). Now consider the map $\hat{B} \to \text{GL}(2)$ defined as the composition

$$\hat{B} \hookrightarrow \hat{P} \twoheadrightarrow \hat{M} \simeq \hat{T}' \times \text{GL}(2) \twoheadrightarrow \text{GL}(2).$$

It maps $\hat{T}$ onto the diagonal torus in GL(2), annihilates all the root groups $\hat{U}_\alpha \lor$ for positive roots $\alpha \neq \beta$, and maps $\hat{U}_{\beta \lor}$ isomorphically to $\left( \begin{array}{cc} 1 & \ast \\ \ast & 1 \end{array} \right)$. The resulting two-dimensional representation $\rho_\beta : \Gamma_{Q_p} \to \text{GL}(2)$, obtained by composing with $\rho$, is of the form $\rho_\beta = \left( \begin{array}{cc} \chi_1 & \ast \\ \chi_2 & \ast \end{array} \right)$ for two potentially crystalline characters $\chi_i : \Gamma_{Q_p} \to E^\times$. Furthermore, by the way we set things up,

$$\chi_1/\chi_2 = \alpha_{\text{GL}(2)} \circ \hat{\chi}_{\rho} = \beta \lor \circ \hat{\chi}_{\rho}.$$

On an open subgroup of $I_{Q_p}$, this character becomes $\epsilon(\chi_{\rho, \text{alg}}, \beta \lor)$, which is a negative power of the cyclotomic character. By 4.2, we infer that $\rho_\beta$ must split. This allows
us to find a $b \in \hat{B}(E)$ such that $(bpb^{-1})_\beta = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$. In particular, $\beta^\vee \notin C_{bpb^{-1}}$. Since $\rho$ is assumed to be good, $C_\rho \subset C_{bpb^{-1}}$, and we can conclude that $\beta^\vee \notin C_\rho$.

**Step 2. The representation $\rho'$ is triangular.**

We show that $\rho' = sps^{-1}$ maps $\Gamma_{Q_\rho}$ into $\hat{B}(E)$. Choose arbitrarily an ordering of the roots in $C_\rho$. As is well-known, the multiplication map gives an isomorphism of varieties,

$$m : \prod_{\alpha^\vee \in C_\rho} \hat{U}_{\alpha^\vee} \xrightarrow{\sim} \hat{U}_{C_\rho},$$

(See (11) on p. 25 in [BH14]). Let $\sigma \in \Gamma_{Q_\rho}$, and find the Jordan decomposition $\rho(\sigma) = tu$, where $t = \hat{\chi}_\rho(\sigma) \in \hat{T}(E)$, and $u = \prod u_{\alpha^\vee} \in \hat{U}_{C_\rho}(E)$. Upon conjugation by $s$, we get $\rho'(\sigma) = t'u'$, where $t' = s\hat{\chi}_\rho(\sigma)s^{-1}$, and $u' = \prod su_{\alpha^\vee}s^{-1}$. It is a general fact that $w\hat{U}_{\alpha^\vee}w^{-1} = \hat{U}_{w(\alpha^\vee)}$ (which we note is used in the proof of 3.2.5 in loc. cit.). Consequently, $su_{\alpha^\vee}s^{-1} \in \hat{U}_{s\beta^\vee(\alpha^\vee)}$. From basic Lie theory, we know that $s\beta^\vee$ permutes $R^{+\vee} - \{\beta^\vee\}$, and sends $\beta^\vee \mapsto -\beta^\vee$. Therefore, to check that $u' \in \hat{U}(E)$, we just have to verify that $\beta^\vee$ does not contribute to the product $u = \prod u_{\alpha^\vee}$. In other words, that $\beta^\vee \notin C_\rho$. This is the preceding Step 1.

**Step 3. $\rho'$ is good, generic, and potentially semistable.**

During Step 2 we found that $\hat{\chi}_{\rho'} = s\beta^\vee(\hat{\chi}_\rho)$. Therefore, $\alpha^\vee \circ \hat{\chi}_{\rho'} = s\beta^\vee(\alpha^\vee) \circ \hat{\chi}_\rho$, for all $\alpha \in R$, which is not among $\{1, \epsilon^{\pm 1}\}$. That is, $\rho'$ is generic (and clearly potentially semistable). $\rho'$ is good by Lemma 3.2.5 in [BH14]: We just have to check that $s\beta^\vee \in WC_\rho$. I.e., that $s\beta^\vee(C_\rho) \subset R^{+\vee}$. Again, this follows from Step 1, since $s\beta^\vee$ permutes $R^{+\vee} - \{\beta^\vee\}$, which contains $C_\rho$.

**Step 4. Invoking the induction hypothesis.**

From the relation $\chi_{\rho',alg} = s\beta(\chi_{\rho,alg})$, we find that $w_{\rho'} = s\beta^\vee w_{\rho}$, which has length $\ell(w_{\rho'}) = \ell(w_{\rho}) - 1$ since $w_{\rho}^{-1}(\beta^\vee)$ is a negative root:

$$\langle \chi_{\rho,alg}^+, w_{\rho}^{-1}(\beta^\vee) \rangle = \langle w_{\rho}^{-1}(\chi_{\rho,alg}), w_{\rho}^{-1}(\beta^\vee) \rangle = \langle \chi_{\rho,alg}, \beta^\vee \rangle < 0.$$

Applying the induction hypothesis to $\rho'$, we infer that $w_{\rho'} \in WC_{\rho'}$. The rest is completely formal, using that $C_{\rho'} = s\beta^\vee(C_\rho)$ (as follows from the computation in Step 2; see also the proof of Lemma 3.3.5 in [BH14]). As a result, $WC_{\rho'} = s\beta^\vee(WC_\rho)$, and we conclude that $w_{\rho} \in WC_{\rho}$, which finishes the proof.

5. Proof of the main result

In conjunction with the results from the previous sections, the following proves the main Theorem 1.2 in the Introduction.
Lemma 5.1. Suppose $\rho: \Gamma_{Q_p} \to \hat{B}(E)$ is good, and satisfies 2.3. Let $V$ be an irreducible algebraic representation of $G(Q_p)$, defined over $E$, of highest weight $\lambda \in X_+(T)$ relative to $B$. Then the following holds.

1. If $\Pi(\rho)^{\text{ord},V-\text{alg}} \neq 0$, then $\chi_\rho$ is locally algebraic, with $\chi_\rho^+ = \lambda + \theta$.
2. If $\chi_\rho$ is locally algebraic, with $\chi_\rho^+ = \lambda + \theta$, then
   \[ \Pi(\rho)^{\text{ord},\text{alg}} = \Pi(\rho)^{\text{ord},V-\text{alg}} = V \otimes_E (\text{Ind}_{B_-}^{G(Q_p)} \chi_{\rho, \text{sm}} \cdot |\theta|^{-1})^{C_w}. \]

Proof. Fix $w \in W_{C_w}$, and choose a composition series $F_j \Pi(\rho)_{C_w, w}$ for $\Pi(\rho)_{C_w, w}$.

\[ 0 \subset F_0 \Pi(\rho)_{C_w, w} \subset F_1 \Pi(\rho)_{C_w, w} \subset \cdots \subset F_j \Pi(\rho)_{C_w, w} \subset \cdots \subset \Pi(\rho)_{C_w, w}. \]

Here, under assumption 2.3, its constituents are given by:

- For $j = 0$,
  \[ F_0 \Pi(\rho)_{C_w, w} = \text{soc}_{G(Q_p)} \Pi(\rho)_{C_w, w} \simeq (\text{Ind}_{B_-}^{G(Q_p)} w^{-1}(\chi_\rho) \cdot (\epsilon^{-1} \circ \theta))^{C_w}. \]

- For $j > 0$,
  \[ \text{gr}_j \Pi(\rho)_{C_w, w} \simeq (\text{Ind}_{B_-}^{G(Q_p)} ((\prod_{\alpha \in I^\vee} s_\alpha) w)^{-1}(\chi_\rho) \cdot (\epsilon^{-1} \circ \theta))^{C_w}, \]

where $I \subset w(S^\vee) \cap C_\rho$ is some non-empty subset of of pairwise orthogonal roots (which of course depends on $j$).

Let $V$ be an irreducible algebraic representation of $G(Q_p)$, defined over $E$, of highest weight $\lambda \in X_+(T)$ relative to $B$. By A.3, we have the exact sequence

\[ 0 \longrightarrow (F_{j-1} \Pi(\rho)_{C_w, w})^{V-\text{alg}} \longrightarrow (F_j \Pi(\rho)_{C_w, w})^{V-\text{alg}} \longrightarrow (\text{gr}_j \Pi(\rho)_{C_w, w})^{V-\text{alg}}. \]

Let us make some initial observations. Suppose $\Pi(\rho)_{C_w, w}^{V-\text{alg}} \neq 0$. Then, for some $j \geq 0$, we must have $(\text{gr}_j \Pi(\rho)_{C_w, w})^{V-\text{alg}} \neq 0$. By part (1) of A.1, we find that the inducing character $((\prod_{\alpha \in I^\vee} s_\alpha) w)^{-1}(\chi_\rho) \cdot (\epsilon^{-1} \circ \theta)$ is locally algebraic, with algebraic part $\lambda$. Therefore, $\chi_\rho$ itself is locally algebraic, and

\[ ((\prod_{\alpha \in I^\vee} s_\alpha) w)^{-1}(\chi_\rho^+) - \theta = \lambda. \]

In particular $((\prod_{\alpha \in I^\vee} s_\alpha) w)^{-1}(\chi_\rho^+) \cdot \theta = \lambda$, where $\chi_\rho^+ = w^{-1}_p(\chi_\rho^+)$ is the dominant $W$-conjugate.

We apply this to $\rho' = w^{-1}_p \rho$, (a good conjugate of $\rho$ by 3.2.5 in [BH14], since $w_\rho \in W_{C_\rho}$). By 3.3.5 in loc. cit. (or rather its proof), we know that there’s an isomorphism $\Pi(\rho)_{C_w, w} \simeq \Pi(\rho')_{C'_w, w'}$, where $C'_\rho = w^{-1}_p(C_\rho)$ and $w' = w^{-1}_p w$. If this has nonzero locally $V$-algebraic vectors, we infer from the previous paragraph that $((\prod_{\alpha \in I^\vee} s_\alpha) w)' = 1$, since $\chi_{\rho', \text{alg}}$ is dominant (by the very definition of $w_\rho$).

Here $I' = w^{-1}_p(I)$ is some subset of orthogonal roots, possibly empty. In fact, we claim $I'$ must be empty. For suppose $I' \neq \emptyset$. Then $\prod_{\alpha \in I'} s_\alpha = w^{-1}_p$ maps $C'_\rho$
into $R^{+,\vee}$, since $w' \in W_{C_{\rho}}$. However, observe that $s_{\alpha}(\beta) = \beta$, for any two distinct $\alpha, \beta \in I'^{\vee}$, since the roots of $I'$ are pairwise orthogonal. Thus $\prod_{\alpha \in I'^{\vee}} s_{\alpha}$ maps any $\alpha \in I'^{\vee}$ to its negative, $-\alpha$. Contradiction. Ergo $w' = 1$, and $I'$ must be empty.

In conclusion, if $\Pi(\rho_{C_{\rho}, w})^{V-\text{alg}} \neq 0$, then $w = w_{\rho}$, and all these $V$-algebraic vectors lie in the socle. Furthermore, $\chi_{\rho}$ must be locally algebraic, with (dominant $W$-conjugate of the) algebraic part $\chi_{\rho, \text{alg}}^{+} = \lambda + \theta$. This proves (1).

For part (2), the first equality follows from (1). To show the second equality, note that, by what was summarized just above,

$$\Pi(\rho)_{C_{\rho}, w}^{\text{ord}, V-\text{alg}} = \Pi(\rho)_{C_{\rho}, w}^{V-\text{alg}} = (\text{soc}_{G(\mathbb{Q}_{p})}\Pi(\rho)_{C_{\rho}, w})^{V-\text{alg}}.$$

Now use the computation in the Appendix; part (2) of A.1. The inducing character $w_{\rho}^{-1}(\chi_{\rho})^{\vee}(\epsilon^{-1} \circ \det)$ has smooth part $w_{\rho}^{-1}(\chi_{\rho, \text{sm}})^{\vee}|\theta|^{-1}$. However, for smooth parabolic induction, we have intertwining operators, which allows us to ignore $w_{\rho}$ here.  

This finishes the proof.

6. An application to local-global compatibility

Chapter 4 of [BH14] presents various local-global compatibility conjectures on how the construction $\Pi(-)^{\text{ord}}$ relates to spaces of $p$-adic and mod $p$ modular forms (see their $p$-adic Conjecture 4.2.2, and the mod $p$ Conjecture 4.2.5). We employ the notation of loc. cit., without much comment, and focus on the $p$-adic case. Basically, one starts off with a Galois representation $r : \Gamma_{F} \to \text{GL}_{n}(E)$, where $F/F^{+}$ is a CM field, which occurs in a space of $p$-adic modular forms $\hat{S}(U_{p}, E)$ on some definite unitary group $G/F^{+}$ split above $p$. Assuming $r_{w} = r|_{\Gamma_{F_{w}}}$ is triangular and generic, for all places $w|_{p}$ of $F$, it is expected that for some integer $d > 0$,

$$\bigotimes_{v|_{p}} \Pi(r_{v})^{\text{ord}} \otimes ((\epsilon^{n-1} \circ \det))^{\otimes d} \iso \hat{S}(U_{p}, E)[p]^{\Sigma}^{\text{ord}},$$

as representations of $G(F^{+} \otimes \mathbb{Q}_{p})$. Here $p^{\Sigma} \subset \mathbb{T}^{\Sigma}$ is the prime ideal of the Hecke algebra associated with $r$, see p. 45 in [BH14]. Theorem 4.4.8 in loc. cit. gives a speck of evidence for 6.1, proving there is an injection under rather strong hypotheses: Notably, they assume $r$ is modular (which is something one would like to deduce from 6.1, cf. Fontaine-Mazur), and they assume all the restrictions $\bar{r}_{w} = \bar{r}|_{\Gamma_{F_{w}}}$ are "totally indecomposable" for $w|_{p}$—that is, $C_{\bar{r}_{w}} = R^{+,\vee}$ is maximal— or in other words, $W_{C_{\bar{r}_{w}}} = 1$, so that $\Pi(r_{v})^{\text{ord}}$ has a simple socle. Proposition 4.1 lets us relax this last hypothesis, by essentially running the same argument, but instead of getting an injection we only get a nonzero map in general. As far as we know, this is the first result towards 6.1 with no restrictions on the sets $C_{\bar{r}_{w}}$. 
Corollary 6.2. Let \( r : \Gamma_F \to GL_n(E) \) be a Galois representation which satisfies the following assumptions:

(a) \( r \) is modular.
(b) \( \bar{r}|_{\Gamma_F(\zeta_p)} \) is absolutely irreducible.
(c) \( r_w \) is triangular and generic, for \( w \mid p \).
(d) \( \bar{r}_w \) is triangular and inertially generic, for \( w \mid p \).
(e) \( p > 2n + 2 \) and \( \zeta_p \notin F \).

Fix \( U_p \) such that \( r \) is modular of (some weight and) level \( U_p U_p \), for some \( U_p \) small enough. Then there is a \( G(F^+ \otimes \mathbb{Q}_p) \)-equivariant nonzero map,

\[
\prod_{v \mid p} \Pi(r_v)^{\text{ord}} \otimes (\epsilon^{n-1} \circ \text{det}) \longrightarrow \hat{S}(U_p^p, E)[p^\Sigma],
\]

in the category of admissible unitary continuous representations.

Remarks 6.3. Since we are not saying anything about surjectivity, we may as well take \( d = 1 \) above. The conditions (a)–(e) can be traced back to the modularity lifting theorems used in [BH14]. The differences with their Theorem 4.4.8 is that our condition (d) is weaker; it does not require \( C_{\bar{r}_w} \) to be maximal – but instead of an injection we only get a nonzero map.

Proof. Pick an automorphic representation \( \pi \) of \( G(A_F^+) \), of level of the form \( U_p U_p \), such that \( r \simeq r_\pi \). In particular, \( r_w \) is potentially crystalline with distinct Hodge-Tate weights, for \( w \mid p \) – as follows from the ordinarity assumption (c). Let \( \widetilde{BS}(r_w) \) be the locally algebraic representation of \( GL_n(F_w) \) attached to \( r_w \) by the Breuil-Schneider recipe, normalized as in [Sor]. Essentially by local-global compatibility at the places above \( p \) (due to Caraiani and others), and base change, we know that

\[
\widetilde{BS}(r) := \prod_{v \mid p} \widetilde{BS}(r_v) \hookrightarrow \hat{S}(U_p^p, E)[p^\Sigma].
\]

(See Chapter 3 of [Sor] for details on how this goes.) On the other hand, since \( r_v \) is ordinary, we can relate \( \widetilde{BS}(r_v) \) to one of the principal series occurring in the socle of \( \Pi(r_v)^{\text{ord}} \). The twist \( \epsilon^{n-1} \circ \text{det} \) arises in the distinction between \( \widetilde{BS}(r_v) \), and the original normalization \( BS(r_v) \) from [BrSc], see Section 2.4 in [Sor], and also 4.2.1 in [BH14]. The latter is what sits in \( \Pi(r_v)^{\text{ord}} \). Indeed, identifying \( F_v = \mathbb{Q}_p \),

\[
BS(r_v) \simeq (\text{Ind}_{B_-^{\mathbb{Q}_p}}^{GL_n(\mathbb{Q}_p)}) w^{-1}_v (\chi_r \cdot (\epsilon^{-1} \circ \theta))^{\text{alg}},
\]

Passing to the universal unitary completion, one gets an injection

\[
\prod_{v \mid p} (\text{Ind}_{B_-^{\mathbb{Q}_p}}^{GL_n(\mathbb{Q}_p)}) w^{-1}_v (\chi_r \otimes \chi_{n-1} \otimes (\epsilon^{-1} \circ \theta))^{\text{alg}} \hookrightarrow \hat{S}(U_p^p, E)[p^\Sigma].
\]
Now use Corollary 4.3.11 in [BH14] for each $\rho = r_\epsilon \otimes \epsilon^{n-1}$, which applies by (a)–(e), getting that restriction to the socle induces an isomorphism

\[ \text{Hom}_{GL_n(\mathbb{Q}_p)}(\Pi(\rho)^{ord}, \hat{S}) \cong \bigoplus_{w \in W_{C_p}} \text{Hom}_{GL_n(\mathbb{Q}_p)}((\text{Ind}_{B^-}^{GL_n(\mathbb{Q}_p)} w^{-1}(\chi_\rho)(\epsilon^{-1} \circ \theta))^{c_0}, \hat{S}), \]

where $\hat{S}$ is short-hand for $\hat{S}(U^p, E)[p^\infty]$. Finally, note that $w_p \in W_{C_p}$ by 4.1. □

**Appendix A. Continuous principal series**

For convenience, we gather a few well-known facts about locally algebraic vectors in continuous principal series. For lack of a reference, we provide proofs. As in the main text, $G/\mathbb{Q}_p$ denotes a split connected reductive group, and $B = TU$ a choice of Borel subgroup. Let $B^- = TU^-$ be the opposite Borel subgroup. Suppose we are given a unitary continuous character $\chi : T(\mathbb{Q}_p) \to \mathcal{O}_E^\times$, with values in a finite extension $E/\mathbb{Q}_p$. We inflate it to $B^-\mathbb{Q}_p)$, by making it trivial on the unipotent radical, and look at the continuous parabolic induction,

\[ I(\chi) = (\text{Ind}_{B^-}^{G(\mathbb{Q}_p)} \chi)^{c_0} = \{ f : G(\mathbb{Q}_p) \xrightarrow{ctx} E, f(bg) = \chi(b)f(g) \}. \]

Endowed with the sup-norm, this becomes a $p$-adic Banach space on which $G(\mathbb{Q}_p)$ acts unitarily via right translations. It affords an admissible unitary continuous representation of $G(\mathbb{Q}_p)$ – which conjecturally is topologically irreducible precisely when $\chi \circ \alpha^\vee \neq 1$ for all simple roots $\alpha$. (Irreducibility is known if this condition holds mod $p$. This is due to Ollivier for GL$_n$, and Abe in general.)

**Lemma A.1.** Let $V$ be an irreducible algebraic representation of $G(\mathbb{Q}_p)$, defined over $E$, of highest weight $\lambda \in X_+(T)$ relative to $B$. Then the following holds.

1. If $I(\chi)^{V-\text{alg}} \neq 0$, then $\chi$ is locally algebraic, with algebraic part $\chi_{\text{alg}} = \lambda$.
2. If $\chi = \chi_{\text{alg}} \cdot \chi_{\text{sm}}$ is locally algebraic, with $\chi_{\text{alg}} = \lambda$, then

\[ I(\chi)^{\text{alg}} = I(\chi)^{V-\text{alg}} = V \otimes_E (\text{Ind}_{B^-}^{G(\mathbb{Q}_p)} \chi_{\text{sm}})^{c_0}. \]

(Here the superscript $c_0$ means we take the smooth induction of $\chi_{\text{sm}}$.)

**Proof.** Recall that the locally $V$-algebraic vectors in $I(\chi)$ are defined as the subspace

\[ I(\chi)^{V-\text{alg}} = \varprojlim_K V \otimes_E \text{Hom}_K(V, I(\chi)) \to I(\chi), \]

with $K$ ranging over the compact open subgroups of $G(\mathbb{Q}_p)$. For each such $K$,

\[ (A.2) \quad \text{Hom}_K(V, I(\chi)) \cong \bigoplus_{x \in B^- \setminus G(\mathbb{Q}_p)/G(\mathbb{Q}_p)} (V^\vee|_{B^- \mathbb{Q}_p} \otimes \chi)^{B^- \mathbb{Q}_p \cap xKx^{-1}}. \]

To show part (1), suppose this is nonzero. Pick an $x$ such that the corresponding summand is nonzero. Thus $V^\vee$ contains a vector $v \neq 0$ on which $B^- \mathbb{Q}_p \cap xKx^{-1}$ acts via (the inflation of) the inverse character $\chi^{-1}$. In particular, the unipotent...
elements $U^-(\mathbb{Q}_p) \cap xKx^{-1}$ must act trivially on $v$. Therefore, since this subgroup is Zariski dense in the full unipotent radical, we deduce that $v \in (V^\vee)^{U^-(\mathbb{Q}_p)}$ must be a highest weight vector (relative to the opposite Borel). Consequently, $B^-(\mathbb{Q}_p)$ acts on $v$ via the highest weight $-\lambda$. We conclude that $\chi$ and $\lambda$ agree on the open subgroup $T(\mathbb{Q}_p) \cap xKx^{-1}$, and therefore $\chi$ is locally algebraic with $\chi_{\text{alg}} = \lambda$.

For part (2), suppose $\chi = \lambda \cdot \chi_{\text{sm}}$. The first equality follows from part (1). For the second equality, we realize $V$ as the algebraically induced representation $\text{Ind}_{B^-}(\mathbb{Q}_p)^\lambda$, and show that the natural multiplication map

$$(\text{Ind}_{B^-}(\mathbb{Q}_p)\chi_{\text{sm}})^\mathbb{K} \to \text{Hom}_K(V, I(\chi))$$

$f_{\text{sm}} \mapsto (f_{\text{alg}} \mapsto f_{\text{alg}} \cdot f_{\text{sm}})$

is an isomorphism for sufficiently small $K$. Tensoring with $V$, and taking the limit $\lim_{\to K}$ then yields the result. First off, the multiplication map is injective: For suppose $f_{\text{alg}} \cdot f_{\text{sm}} = 0$, for all $f_{\text{alg}}$. Let $f_+ \in V$ be a highest weight vector, and take $f_{\text{alg}} = g^{-1} \cdot f_+$, for varying $g \in G(\mathbb{Q}_p)$. Since $f_+(e) \neq 0$, this shows $f_{\text{sm}}(g) = 0$. To get bijectivity, we compare dimensions. Taking $K$ small enough that $\chi_{\text{sm}} = 1$ on every subgroup $B^-(\mathbb{Q}_p) \cap xKx^{-1}$, as in the previous paragraph, the source is identified with $E^\#B^-(\mathbb{Q}_p) \setminus G(\mathbb{Q}_p)/\mathbb{K}$. It remains to show the target has the same dimension. In turn, in A.2 we must show that each summand $(V^\vee|_{B^-}(\mathbb{Q}_p) \otimes \chi)^{B^-(\mathbb{Q}_p) \cap xKx^{-1}}$ is (at most) one-dimensional. We break up $V^\vee$ into weight spaces, as $\oplus_{\mu \in X(T)} V^\vee(\mu)$.

$$(V^\vee|_{B^-}(\mathbb{Q}_p) \otimes \chi)^{B^-(\mathbb{Q}_p) \cap xKx^{-1}} \hookrightarrow \bigoplus_{\mu \in X(T)} (V^\vee(\mu) \otimes \chi)^{T(\mathbb{Q}_p) \cap xKx^{-1}}.$$

Only the highest weight $-\lambda$ contributes. Indeed, if the $\mu$-summand is nontrivial, we must have $\mu = -\lambda$ (since $\chi_{\text{alg}} = \lambda$ and $\chi_{\text{sm}}$ is trivial on every $T(\mathbb{Q}_p) \cap xKx^{-1}$). Finally, reminding ourselves that $\dim_E V^\vee(-\lambda) = 1$, concludes the proof. $\square$

In the main body of the text we have used the following basic observation repeatedly, often without mention.

**Lemma A.3.** The functors $(-)^{\text{alg}}$ and $(-)^{V-\text{alg}}$ are left exact on the category of $G(\mathbb{Q}_p)$-representations over $E$.

**Proof.** Say we are given an exact sequence of $G(\mathbb{Q}_p)$-representations over $E$,

$$0 \to W' \to W \to W'' \to 0.$$

Apply $\text{Hom}_K(V, -)$, then tensor with $V$ over $E$, and take the limit over $K$. This results in a long exact sequence,

$$0 \to W'^{V-\text{alg}} \to W^{V-\text{alg}} \to W''^{V-\text{alg}} \to \lim_{\to K} V \otimes_E \text{Ext}_K^1(V, W') \to \cdots,$$

which proves left exactness of $(-)^{V-\text{alg}}$. Now use that $W^{\text{alg}} = \oplus V W'^{V-\text{alg}}$. $\square$
References


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