Coordinates and Change of Basis

Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be an ordered basis for the $n$-dimensional vector space $V$.

If $v$ is any vector in $V$, then there is a unique coordinate vector $[v]_B = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathbb{R}^n$ such that $v = x_1 b_1 + \cdots + x_n b_n = [b_1, \ldots, b_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \mathcal{B}[v]_B$.

If $\mathcal{D} = \{d_1, \ldots, d_n\}$ is another ordered basis for $V$, then there are unique scalars $P_{ij}$ such that $d_j = \sum_{i=1}^{n} b_i P_{ij}$, for $1 \leq j \leq n$. In other words,

$$[d_1, \ldots, d_n] = [b_1, \ldots, b_n] \begin{bmatrix} P_{11} & \cdots & P_{1n} \\ \vdots & \ddots & \vdots \\ P_{n1} & \cdots & P_{nn} \end{bmatrix}.$$  

Thus $\mathcal{D} = \mathcal{B} P$, where $P$ is the matrix $(P_{ij})$.

Let $[v]_D = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$. Then $v = y_1 d_1 + \cdots + y_n d_n = [d_1, \ldots, d_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \mathcal{D}[v]_D$.

Since $v$ is also equal to $\mathcal{B}[v]_B$, we have $\mathcal{D}[v]_D = \mathcal{B}[v]_B$, and since $\mathcal{D} = \mathcal{B} P$, it follows that $\mathcal{B} P [v]_D = \mathcal{B}[v]_B$.

Since $\mathcal{B}$ is a basis for $V$, $v$ is uniquely represented as a linear combination of elements of $\mathcal{B}$. This means that

$$[v]_B = P [v]_D.$$ 

Similarly,

$$[v]_D = P^{-1} [v]_B.$$ 

$P$ is called the transition matrix from the ordered basis $\mathcal{D}$ to the ordered basis $\mathcal{B}$.

**Note:** In the notation of **Proposition and definition 2.6.18** on page 217 of your book,

$$P = [P_{\mathcal{D} \rightarrow \mathcal{B}}].$$ 

It’s also worth observing that

$$P^{-1} = [P_{\mathcal{D} \rightarrow \mathcal{B}}]^{-1} = [P_{\mathcal{B} \rightarrow \mathcal{D}}].$$