Transition Matrix Exercises

1. Recall the double-angle trigonometric identities.

\[ \cos^2(x) - \sin^2(x) = \cos(2x). \]
\[ 2 \sin(x) \cos(x) = \sin(2x). \]

(a) Show that \( S = \{\sin^2(x), \cos^2(x), \sin(x) \cos(x)\} \) and \( T = \{1, \sin(2x), \cos(2x)\} \) span the same 3-dimensional subspace of \( C[0, \pi] = \{f : [0, \pi] \to \mathbb{R} \mid f \text{ is continuous}\} \).

(b) Find the transition matrix from the ordered basis \( S \) to the ordered basis \( T \).

(c) Use the transition matrix to express \( a \sin^2(x) + b \cos^2(x) \) as a linear combination of 1, \( \sin(2x) \) and \( \cos(2x) \).

2. Recall that the hyperbolic cosine and hyperbolic sine functions are defined as follows.

\[ \cosh(x) = \frac{e^x + e^{-x}}{2}, \]
\[ \sinh(x) = \frac{e^x - e^{-x}}{2}. \]

(a) Show that \( E = \{e^x, e^{-x}\} \) and \( H = \{\cosh(x), \sinh(x)\} \) span the same 2-dimensional subspace of \( C(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\} \).

(b) Find the transition matrix from the ordered basis \( E \) to the ordered basis \( H \).

(c) Use the transition matrix to express \( ce^x + de^{-x} \) as a linear combination of \( \sinh(x) \) and \( \cosh(x) \).

3. Using the definition of the hyperbolic cosine function and the hyperbolic sine function, it is a straightforward computation to verify the following identities.

\[ \cosh^2(x) - \sinh^2(x) = 1. \]
\[ \cosh^2(x) + \sinh^2(x) = \cosh(2x). \]
\[ 2 \cosh(x) \sinh(x) = \sinh(2x). \]

(a) Show that \( S_h = \{\cosh^2(x), \sinh^2(x), \cosh(x) \sinh(x)\} \) and \( T_h = \{1, \cosh(2x), \sinh(2x)\} \) span the same 3-dimensional subspace of \( C(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\} \).

(b) Find the transition matrix from the ordered basis \( S_h \) to the ordered basis \( T_h \).

(c) Use the transition matrix to express \( a \cosh^2(x) + b \sinh^2(x) \) as a linear combination of 1, \( \cosh(2x) \) and \( \sinh(2x) \).
1. (b) $P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$.

(c) $a \sin^2(x) + b \cos^2(x) = \frac{1}{2}(a + b) + \frac{1}{2}(-a + b) \cos(2x)$.

2. (b) $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

(c) $ce^x + de^x = (c + d) \cosh(x) + (c - d) \sinh(x)$.

3. (b) $P = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$.

(c) $a \cosh^2(x) + b \sinh^2(x) = \frac{1}{2}(a - b) + \frac{1}{2}(a + b) \cosh(2x)$.