1. [10 points total]: Circle your answer to each of the following true/false or multiple-choice questions. [2 points each]:

Let \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a function and \( \vec{x}_0 \in \mathbb{R}^n \).

(a) **True** or **False**: If \( f \) is differentiable at \( \vec{x}_0 \), then all of the partial derivatives of \( f \) must exist at \( \vec{x}_0 \).

(b) **True** or **False**: If all of the partial derivatives of \( f \) exist at \( \vec{x}_0 \), then \( f \) must be differentiable at \( \vec{x}_0 \).

(c) **True** or **False**: If \( f \) is differentiable at \( \vec{x}_0 \), then \( f \) must be continuous at \( \vec{x}_0 \).

(d) Which of the following double integrals are evaluated over this shaded domain?

\[
\begin{align*}
(i) & \quad \int_{0}^{1} \int_{x^2}^{\sqrt{x}} dy \, dx \\
(ii) & \quad \int_{0}^{1} \int_{y^2}^{\sqrt{y}} dx \, dy \\
(iii) & \quad \int_{0}^{\sqrt{y}} \int_{0}^{y^2} dx \, dy
\end{align*}
\]

A. (i) only \\
B. (ii) only \\
C. (iii) only \\
D. (i) and (ii) \\
E. (i) and (iii)

(e) The shaded region below is

A. \( x \)-simple \\
B. \( y \)-simple \\
C. both \( x \)-simple and \( y \)-simple \\
D. neither \( x \)-simple nor \( y \)-simple
2. [10 points total]:

(a) [2 points]: Let \( g : \mathbb{R}^3 \rightarrow \mathbb{R} \) be defined by \( g(x, y, z) = xy^2 + \cos z \). Find the matrix of partial derivatives \( Dg(x, y, z) \).

\[
Solution:
Dg(x, y, z) = \begin{bmatrix} y^2 & 2xy & -\sin z \end{bmatrix}
\]

(b) [3 points]: Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) be defined by \( f(u, v) = (u, v \sin u, e^v - u) \). Find the matrix of partial derivatives \( Df(u, v) \).

\[
Solution:
Df(u, v) = \begin{bmatrix} 1 & 0 & v \cos u & 0 & 0 & 0 \\
0 & \sin u & 0 & 0 & 0 & 0 \\
-e^v - u & e^v - u & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
(c) [2 points]: Find $f(0,1)$ and $Df(0,1)$.

**Solution:**

\[
f(0,1) = (0,1 \cdot \sin 0, e^{1-0}) = (0,0,e)
\]

\[
Df(0,1) = \begin{bmatrix}
1 & 0 \\
(1) \cos 0 & \sin 0 \\
-e^{1-0} & e^{1-0}
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
1 & 0 \\
-e & e
\end{bmatrix}
\]

(d) [3 points]: Let $h : \mathbb{R}^3 \to \mathbb{R}$ be defined by $h = g \circ f$. Use the chain rule to find $Dh(0,1)$.

**Solution:** By the chain rule,

\[
Dh(0,1) = [Dg(f(0,1))][Df(0,1)] = [Dg(0,0,e)]\begin{bmatrix}
1 & 0 \\
1 & 0 \\
-e & e
\end{bmatrix}
\]

We have

\[
[Dg(0,0,e)] = \begin{bmatrix} 0^2 & 2(0)(0) & -\sin e \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\sin e \end{bmatrix}
\]

and hence

\[
Dh(0,1) = \begin{bmatrix} 0 & 0 & -\sin e \end{bmatrix}\begin{bmatrix} 1 & 0 \\
1 & 0 \\
-e & e
\end{bmatrix}
\]

\[
= \begin{bmatrix} 0(1) + (0)(1) + (-\sin e)(-e) \\
(0)(0) + (0)(0) + (-\sin e)(e) \\
e \sin e & -e \sin e
\end{bmatrix}
\]
3. [10 points]: Evaluate 
\[ \int_e^1 \int_{\ln x}^1 \frac{\cos(y^2)}{x} \, dy \, dx \]

*Solution:* Since we don’t know an easy \( y \)-antiderivative for \( \frac{\cos(y^2)}{x} \), we want to switch the order of integration. To do so, we draw the region of integration:

\[
\int_e^1 \int_{\ln x}^1 \frac{\cos(y^2)}{x} \, dy \, dx = \int_0^1 \int_{\ln(e^y)}^{e^y} \frac{\cos(y^2)}{x} \, dx \, dy
\]

Then
\[
\int_0^1 \int_{\ln(e^y)}^{e^y} \frac{\cos(y^2)}{x} \, dx \, dy = \int_0^1 \cos(y^2) \left[ \ln x \right]_1^{e^y} \, dy
\]
\[
= \int_0^1 \cos(y^2) \left[ \ln(e^y) - \ln 1 \right] \, dy
\]
\[
= \int_0^1 \cos(y^2) \left[ y - 0 \right] \, dy
\]
\[
= \int_0^1 y \cos(y^2) \, dy
\]
\[
= \left[ \frac{\sin y^2}{2} \right]_0^1
\]
\[
= \left[ \frac{\sin(1)}{2} \right] - \left[ \frac{\sin 0}{2} \right]
\]
\[
= \frac{\sin(1)}{2}
\]
4. [10 points]: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(u, v) = (u + e^v, -2u + e^v)$. Let $D^* = [0, 1] \times [0, 1]$ in the $u$-$v$ plane. Calculate

$$\iint_{T(D^*)} [x - y] \, dA$$

**Solution:** Since $DT(u, v) = \begin{bmatrix} 1 & e^v \\ -2 & e^v \end{bmatrix}$ we see that the partial derivatives of $T$ all exist and are continuous. Moreover, $T$ is one-to-one on $D^*$ since if $x = u + e^v$ and $y = -2u + e^v$ for $(u, v) \in D^*$, then $u = \frac{x - y}{3}$ and $v = \ln \frac{2x + y}{3}$. Thus $T$ is one-to-one and continuously differentiable, so by the change of variables theorem,

$$\iint_{T(D^*)} f(x, y) \, dA = \iint_{D^*} f(u + e^v, -2u + e^v) \cdot |\det DT(u, v)| \, dA$$

for any integrable function $f : \mathbb{R}^2 \to \mathbb{R}$. We have

$$\det DT(u, v) = \begin{vmatrix} 1 & e^v \\ -2 & e^v \end{vmatrix} = (1)(e^v) - (-2)(e^v) = 3e^v$$

and so taking $f(x, y) = x - y$ we have

$$\iint_{T(D^*)} x - y \, dA = \iint_{D^*} ((u + e^v) - (-2u + e^v))|3e^v| \, dA$$

$$= \int_0^1 \int_0^1 (3u)(3e^v) \, dv \, du$$

$$= \int_0^1 9u [e^v]_0^1 \, du$$

$$= \int_0^1 9u [e^1 - e^0] \, du$$

$$= 9(e - 1) \left[ \frac{u^2}{2} \right]_0^1$$

$$= \frac{9}{2}(e - 1)$$
5. [10 points]: One day, Bill the baker spills an entire bag of sugar on his (2 meter)-by-(2 meter) baking table, represented by \( D = [-1, 1] \times [-1, 1] \). Assume the planar density of sugar over a point \((x, y)\) is given by

\[
f(x, y) = x^2 + y^2 \text{ mg/m}^2
\]

Luke the very lucky ant then eats his way across the table in path given by

\[
c(t) = (e^{-t} \cos t, e^{-t} \sin t) \quad \text{from} \quad t = 0 \quad \text{to} \quad t = 10.
\]

If the path Luke eats is 1 millimeter \((= \frac{1}{1000} \text{ meters})\) wide, then

\[
[f(c(t))] \text{ mg/m}^2 \left[ \frac{1}{1000} \text{ m} \right] = \frac{f(c(t))}{1000} \text{ mg/m}
\]

approximates the linear density of sugar along Luke’s path. In mg, about how much sugar did Luke eat?

\[
\text{Solution: Since} \quad \frac{f(c(t))}{1000} \text{ approximates the linear density of sugar along the path} \ c, \ \text{the total mass of sugar eaten (in mg) is given by} \quad \int_c \frac{f(x,y)}{1000} \, ds = \frac{1}{1000} \int_0^{10} f(c(t)) ||c'(t)|| \, dt.
\]

For \(0 \leq t \leq 10\) we have

\[
f(c(t)) = e^{-2t} \cos^2 t + e^{-2t} \sin^2 t = e^{-2t} \quad \text{(using} \sin^2 t + \cos^2 t = 1)\]

and

\[
||c'(t)|| = ||(-e^{-t} \cos t + e^{-t}(- \sin t), -e^{-t} \sin t + e^{-t} \cos t)||
\]

\[
= \sqrt{(-e^{-t} \cos t - e^{-t} \sin t)^2 + (-e^{-t} \sin t + e^{-t} \cos t)^2}
\]

\[
= \sqrt{e^{-2t} \cos^2 t + 2(e^{-t} \cos t)(e^{-t} \sin t) + e^{-2t} \sin^2 t + e^{-2t} \sin^2 t t - 2(e^{-t} \sin t)(e^{-t} \cos t) + e^{-2t} \cos^2 t}
\]

\[
= \sqrt{2e^{-2t}} \quad \text{(using} \sin^2 t + \cos^2 t = 1)\]

Hence

\[
\frac{1}{1000} \int_0^{10} f(c(t)) ||c'(t)|| \, dt = \frac{1}{1000} \int_0^{10} (e^{-2t})(\sqrt{2} e^{-t}) \, dt
\]

\[
= \frac{\sqrt{2}}{1000} \int_0^{10} e^{-3t} \, dt
\]

\[
= \frac{\sqrt{2}}{1000} \left[ -\frac{e^{-3t}}{3} \right]_0^{10}
\]

\[
= \frac{\sqrt{2}}{1000} \left[ -\frac{e^{-30}}{3} - \frac{-e^0}{3} \right]
\]

\[
= \frac{\sqrt{2}}{3000} (1 - e^{-30})
\]

so Luke ate \(\frac{\sqrt{2}}{3000} (1 - e^{-30})\) mg of sugar. That must have been a small bag of sugar!