1. Let \( \vec{F}(x, y, z) = (2 \cos(y^2), -4xy \sin(y^2) + e^z, ye^z) \) for \((x, y, z) \in \mathbb{R}^3\).

(a) Find a function \( f : \mathbb{R}^3 \to \mathbb{R} \) such that \( \vec{F} = \nabla f \).

(b) Let \( c \) be the piecewise linear path in \( \mathbb{R}^3 \) traveling from \((2, 1, 5)\) to \((1, 3, 4)\) to \((0, 1, 0)\), in that order. Evaluate \( \int_c \vec{F} \cdot d\vec{s} \).

2. Suppose that in the year 2020 I become a sea captain, and the bottom of my (huge) ship is parametrized by

\[
\Phi(\theta, \phi) = (\sin \phi \cos \theta, 5 \cos \phi, \sin \phi \sin \theta)
\]

for \((\theta, \phi)\) in the domain \( D = [\pi, 2\pi] \times [0, \pi] \), with lengths measured in kilometers (km).

(a) Calculate \( \| \vec{T}_\theta \times \vec{T}_\phi \| \).

(b) Over the years, barnacles and other debris become attached to the bottom of the ship with density given by \( f(x, y, z) = |y| \) tonnes/km\(^2\). What is the total mass (in tonnes) of debris on the bottom of the ship?

3. Let \( S \) be the portion of the paraboloid \( z + x^2 + y^2 = 1 \) in the first octant (where \( x \geq 0, y \geq 0, z \geq 0 \)), oriented by the outward pointing normal as pictured.

(a) Consider the parametrization of \( S \) given by

\[
\Phi(z, \theta) = ((1 - z)^{1/2} \cos \theta, (1 - z)^{1/2} \sin \theta, z)
\]

for \((z, \theta)\) in the domain \( D = [0, 1] \times [0, \pi/2] \). Is \( \Phi \) an orientation preserving parametrization of \( S \)?

(b) Suppose the temperature at a point \((x, y, z) \in S\) is given by \( T(x, y, z) = xyz \). Find the heat flow vector field \( \vec{F} = -k \nabla T \) in terms of the heat constant \( k \).

(c) If we take \( k = 1 \), what is the heat flux across the oriented surface \( S \)?
4. Let $S$ be the ellipsoid $x^2 + 2y^2 + 2z^2 = 1$ which can be parametrized by a scaled analogue of spherical coordinates:

$$
\Phi(\theta, \phi) = (\sin \phi \cos \theta, \frac{1}{\sqrt{2}} \sin \phi \sin \theta, \frac{1}{\sqrt{2}} \cos \phi)
$$

for $(\theta, \phi)$ in the domain $D = [0, 2\pi] \times [0, \pi]$.

(a) Find the tangent plane to $S$ at the point $(\frac{1}{\sqrt{2}}, \frac{1}{2}, 0)$.

(b) Write down (but do not evaluate) an iterated integral for the surface area of $S$.

5. Circle your answer to each of the following true/false or multiple-choice questions:

(a) Which of the following pictures represents the vector field $\vec{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $\vec{F}(x, y) = (yx, -x)$?

(b) Recall that a sphere of radius $R$ can be parametrized by $\Phi(\theta, \phi) = (R \sin \phi \cos \theta, R \sin \phi \sin \theta, R \cos \phi)$ for $(\theta, \phi)$ in $D = [0, 2\pi] \times [0, \pi]$. Which of the following pictures shows $S = \Phi(D)$ for $\Phi(\theta, \phi) = ( (15 + \cos 15\phi) \sin \phi \cos \theta, (15 + \cos 15\phi) \sin \phi \sin \theta, (15 + \cos 15\phi) \cos \phi )$?

(c) Which of the following is an equation for the pictured surface?

A. $x^2 - y^2 + z = 0$

B. $x^2 - y + z^2 = 0$

C. $x^2 - y^4 + z^2 = 0$
6. The equation \( x^2 + y^2 = (2 + \cos z)^2 \) determines the pictured surface \( S \) for \( 0 \leq z \leq 3\pi \). Find a domain \( D \subset \mathbb{R}^2 \) and a parametrization \( \Phi : D \to \mathbb{R}^3 \) such that \( S = \Phi(D) \).

7. Let \( D = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, 1 \leq x^2 + y^2 \leq 4\} \).

(a) Draw a picture of \( D \) and draw arrows on the boundary \( \partial D \) to indicate its orientation.

(b) Calculate \( \iint_D 4xy \, dA \) using polar coordinates.

(c) Let \( P(x, y) = -x(x^2 + y^2 - 4) \) and \( Q(x, y) = y(x^2 + y^2 - 4) \). Calculate \( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \).

(d) Let \( c_1 \) denote the part of the circle \( x^2 + y^2 = 4 \) lying in the first quadrant. Show that \( \int_{c_1} Pdx + Qdy = 0 \).

(e) Let \( c_2 \) denote the part of \( \partial D \) not covered by \( c_1 \), i.e. the segment \([1, 2]\) in the \( y \)-axis, the part of the circle \( x^2 + y^2 = 1 \) in the first quadrant, and the segment \([1, 2]\) in the \( x \)-axis, oriented as in your picture.

Use Green’s Theorem (along with parts (b)-(d)) to evaluate

\[
\int_{c_2} -x(x^2 + y^2 - 4)dx + y(x^2 + y^2 - 4)dy
\]