Math 10C, Lecture 5

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2016-08-10
First things first

- The midterm is on Friday! Bring
  - a Blue Book and something to write with;
  - no notes, calculators, or other electronic devices.
- Homework 3 is now due at 5:00pm today
- Homework 4 is due on Friday at noon; homework 4 will be shorter than homework 3 so that you can still study for the midterm.
Consider the following statements:

1. Knowing all of the first-order partial derivatives of a function $f$ at a point is the same as knowing all the directional derivatives of $f$ at that point.

2. The gradient of a function at a point is tangent to the level set of the function containing that point.

Which are true?

(A) Both
(B) Only 1
(C) Only 2
(D) Neither
Partial derivatives are (a special kind of) directional derivatives

► Try it: let

\[ f(x_1, x_2, x_3, x_4, x_5) = x_1 \cos(x_2) + \log(x_3^2 + 1) + x_4 e^{x_5} \]

and \( \vec{v} = \langle 1, 0, 0, 0, 0 \rangle \).

What is \( D_{\vec{v}}(f)(1, \pi, 3, -7, 0) \)?

► What if, instead, \( \vec{v} = \langle 0, 1, 0, 0, 0 \rangle \)?

► More generally, if \( \vec{v} \) is the vector with 1 in the \( i \)-th place and 0 everywhere else, when we take the dot product with \( \nabla f(x_1, x_2, \ldots, x_n) \), we’re left with \( \frac{\partial f}{\partial x_i}(x_1, x_2, \ldots, x_n) \).

► So, if we know all directional derivatives, we know all partial derivatives; if we know all partial derivatives, we know all directional derivatives. They’re the same!
Interpreting directional derivatives, I

Example

Alice is an astronaut exploring a new world. Her suit has a limited ability to indicate how temperature is changing in various directions from her current position. The display on her suit reads:

\[
\frac{\partial T}{\partial x} = 0.07 \quad \frac{\partial T}{\partial y} = 1.2.
\]

Her suit also warns her that her current surroundings are too hot, and that she needs to go to a cooler place quickly.

1. Does Alice find herself in a better or worse situation if she heads in the direction \(\langle 1, -1 \rangle\)?

2. Is it better for Alice to head in the direction \(\langle 1, -1 \rangle\) or \(\langle 1, -2 \rangle\)?
The gradient’s magnitude and direction

**Theorem**

*The gradient of $f$ at the point $(x_1, x_2, \ldots, x_n)$, $\nabla f(x_1, x_2, \ldots, x_n)$,*

1. *has magnitude equal to the maximum value of* $D_{\vec{v}}(f)(x_1, x_2, \ldots, x_n)$

2. *points in the direction of maximum increase of $f$ at the point $(x_1, x_2, \ldots, x_n)$.*

We got this by observing that

$$D_{\vec{v}}(f)(x_1, x_2, \ldots, x_n) = ||\nabla f(x_1, x_2, \ldots, x_n)|| \cos \theta,$$

where $\theta$ is the angle between $\vec{v}$ and $\nabla f(x_1, x_2, \ldots, x_n)$. 
How do we use that theorem?

- $\vec{v}$ points in the same direction as $\vec{\nabla}(f)(x_1, x_2, \ldots, x_n)$ if and only if $D_{\vec{v}}(f)(x_1, x_2, \ldots, x_n) = \|\vec{\nabla}(f)(x_1, x_2, \ldots, x_n)\|$. 

- $D_{\vec{w}}(f)(x_1, x_2, \ldots, x_n) = 0$ if and only if $\vec{u}$ is perpendicular to $\vec{\nabla}(f)(x_1, x_2, \ldots, x_n)$. 

- Homework problem: if $\vec{w} = \langle 1, -1 \rangle$ and $D_{\vec{w}}(h)(5, 9) = \|\vec{\nabla}(h)(5, 9)\|$, then $\vec{w}$ points in the same direction as $\vec{\nabla}(h)(5, 9)$.

  - So if $\vec{u}$ is perpendicular to $\vec{w}$, then $\vec{u}$ is perpendicular to $\vec{\nabla}(h)(5, 9)$, which means $D_{\vec{u}}(h)(5, 9) = 0$. 

  - In particular, since $\vec{u} = \langle 1, 1 \rangle$ is perpendicular to $\vec{w}$, the statement is true.
Which direction should Alice go?

Example

Alice is an astronaut exploring a new world. Her suit has a limited ability to indicate how temperature is changing in various directions from her current position. The display on her suit reads:

\[ \frac{\partial T}{\partial x} = 0.07 \quad \frac{\partial T}{\partial y} = 1.2. \]

Her suit also warns her that her current surroundings are too hot, and that she needs to go to a cooler place quickly.

- \[ \nabla (T) = \langle 0.07, 1.2 \rangle. \]
- She wants temperature to decrease as fast as possible; so she should go in the direction opposite of the gradient
- \[ \implies \langle -0.07, -1.2 \rangle. \]
Interpreting directional derivatives, II

Example
Rosamond walks in a strange wilderness, where the height of the landscape is given by

\[ h(x, y) = (3x + 2) \sin(\pi y) + \cos(\pi x), \]

where \( x \) and \( y \) represent kilometers east and north of her current position.

- What elevation is Rosamond at right now?
- What is \( \nabla f(0, 0) \)? Will Rosamond be going uphill or downhill if she walks in the direction \( \langle 3, -4 \rangle \)?
- Which direction should Rosamond walk if she wants to travel uphill as fast as possible?
- After walking 3km east and 4km south, which direction should Rosamond walk if she wants no change in elevation?
Level set (aka contour) diagrams

- Each colored line is (part of) a level set, called a *contour*
- Typical real-world example: topographical map, weather map
Another quick example of contour diagrams

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[Diagram showing contour lines with points labeled A, B, and C.]