Math 100b Winter 2010 Homework 5

Due 2/19/09 in class, or by 5pm in HW box on 6th floor of AP&M

Reading

All references will be to Beachy and Blair, 3rd edition.

Read 9.1-9.3.

Assigned Problems

Write up neat solutions to these problems.

Section 5.4: 2, 4.
Section 9.1: 1, 13, 14.

Additional Problems

Before the problems, we discuss some setup. Let \( d \) be an integer with \( d \neq 0, d \neq 1 \) and such that \( d \) is squarefree (not divisible by the square of a prime integer.)

Define \( R = \mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d}\} \subseteq \mathbb{C} \). You should verify for yourself that \( R \) is a subring of \( \mathbb{C} \). Note that if \( d \) is positive, then \( R \) is contained in \( \mathbb{R} \), but if \( d \) is negative then the elements of \( R \) are generally complex. The rings \( R \) have many applications in number theory.

Fix \( d \), and consider the ring \( R = \mathbb{Z}[\sqrt{d}] \). If \( r = a + b\sqrt{d} \in R \), we define the norm function \( \delta(r) = a^2 - db^2 \); note that \( \delta(r) \) is always an integer. If \( d \) is negative, \( \delta(r) \) is the square of the complex norm of \( r \); but there is no such description if \( d \) is positive.

1. Let \( R = \mathbb{Z}[\sqrt{d}] \) as above.
   
   (a). Prove that the norm function \( \delta \) is multiplicative: for all \( s, t \in R \), \( \delta(st) = \delta(s)\delta(t) \).
   
   (b). Prove that \( s \in R \) is a unit in \( R \) if and only if \( \delta(s) = \pm 1 \). (Hint: for the direction \( \delta(s) = \pm 1 \) implies \( s \) is a unit, if \( s = a + b\sqrt{d} \) consider \( t = a - b\sqrt{d} \). Using this, prove that if \( d \leq -2 \), then the only units of \( \mathbb{Z}[\sqrt{d}] \) are \( \pm 1 \).
   
   (c). Prove that if \( s \in R \) and \( \delta(s) \) is a prime number in \( \mathbb{Z} \), then \( s \) is an irreducible element of \( R \).
Remark. The rings $R[\sqrt{d}]$ with $d \geq 2$ have infinitely many units, in contrast to the result of part (b) for negative $d$. You saw an example of this in exercises 9.1 13, 14 above.

2. Take $d = -2$, so that $R = \mathbb{Z}[-\sqrt{2}]$. Prove that $R$ is a Euclidean domain with respect to the norm function $\delta(a + b\sqrt{-2}) = a^2 + 2b^2$ defined above.

(Hint: follow carefully the proof we gave (or the book gives) that the Gaussian integers $\mathbb{Z}[\sqrt{-1}]$ is a Euclidean domain.)

Remark: you now know that $\mathbb{Z}[\sqrt{d}]$ is a Euclidean domain with respect to the norm function $\delta$ defined above, for $d = -1, -2$. Actually the rings $\mathbb{Z}[\sqrt{d}]$ are only Euclidean domains for a relatively few small values of $d$.

3. Prove that $R = \mathbb{Z}[\sqrt{-6}]$ is not a UFD. (Hint: See Example 9.2.1 in the text for a similar example. Consider the two factorizations $-6 = (-2)(3) = -\sqrt{6}\sqrt{6},$ and prove that these are two essentially different factorizations into irreducibles, violating the definition of a UFD. To understand what elements are associates of each other in this ring, remember problem 1(b).)

4. Consider $R = \mathbb{Z}[\sqrt{2}]$. Show that the field of fractions $Q(R)$ is isomorphic to $\mathbb{Q}[\sqrt{2}] = \{p + q\sqrt{2} | p, q \in \mathbb{Q}\}$.

(Hint. Define a homomorphism $\phi : Q(R) \to \mathbb{Q}[\sqrt{2}]$ by the formula $[x, y] \mapsto xy^{-1}$, where $[x, y]$ is an arbitrary element of $Q(R)$ (so $x, y \in R$ and $y \neq 0$) in the bracket notation we used for elements of $Q(R)$. You must show that $\phi$ is well-defined! So you need to explain why $xy^{-1}$ is independent of the choice of representative $[x, y]$ of the equivalence class; also, why is $xy^{-1}$ an element of $\mathbb{Q}[\sqrt{2}]$? Once you have shown that $\phi$ is well-defined, show that $\phi$ is a homomorphism of rings, and a bijection.)