Math 200c Spring 2015 Homework 1

Due 4/10/2015 in class or under Rob Won’s office door by 3pm

**Reading assignment:** Read Chapters 1-2 of Atiyah-Macdonald and begin to read Chapter 3. We have covered a good portion of the material in Chapters 1-2 in previous quarters, especially Chapter 2; if you are new to the course you will want to read Chapter 2 more carefully. I will type out all exercises below, even if they come from Atiyah-Macdonald. All rings $R$ are commutative with 1 and you may freely assume that $R$ is not the zero ring to avoid trivialities.

**Assigned problems (all to be turned in).**

1. A ring $R$ is *Boolean* if $x^2 = x$ for all $x \in R$.
   
   (a) For every prime ideal $P$ of a Boolean ring $R$, $R/P \cong \mathbb{F}_2$, where $\mathbb{F}_2$ is the field with two elements.

   (b) Every finitely generated ideal of a Boolean ring $R$ is principal.

   (c) $R$ is Boolean if and only if it is isomorphic to a subring of some direct product of rings $\prod_{\alpha \in A} \mathbb{F}_2$ over some index set $A$.

2. Let $R$ be a ring. Consider the polynomial ring $R[x]$. Let $f = a_0 + a_1 x + \cdots + a_n x^n \in R[x]$. Let $N$ be the nilradical (= prime radical) of $R$.

   (a) Prove that $f \in R[x]$ is nilpotent if and only $a_0, a_1, \ldots, a_n \in N$.

   (b) Prove that $f \in R[x]$ is a unit if and only if $a_0$ is a unit in $R$ and $a_1, \ldots, a_n \in N$. (Hint: a sum of a unit and a nilpotent element in a ring is a unit, by a 200a exercise; assume this result. Note that if $f$ is a unit, its image in $(R/P)[x]$ is also a unit for every prime ideal $P$ of $R$.)

   (c) The prime radical and Jacobson radical of $R[x]$ are equal, and both equal to $N[x]$, the set of polynomials with all of their coefficients in $N$. 

1
3. Let $R$ be a (nonzero) ring and let $R[[x]]$ be the ring of formal power series $f = \sum_{n=0}^{\infty} a_n x^n$ with coefficients in $R$. In the fall you proved in an exercise that $f$ is a unit in $R[[x]]$ if and only if $a_0$ is a unit in $R$; if you are new to the course you should prove this, but don’t write it up.

(a) Prove that if $f$ is nilpotent, then $a_n$ is nilpotent in $R$ for all $n \geq 0$.

(b) If $R$ is noetherian, show that $f$ is nilpotent if and only if $a_n$ is nilpotent in $R$ for all $n \geq 0$.

(c) Prove that $f$ is in the Jacobson radical of $R[[x]]$ if and only if $a_0$ is in the Jacobson radical of $R$. Conclude that the prime radical and Jacobson radical of $R[[x]]$ are never the same.

4. In a ring $R$, let $\Sigma$ be the set of all ideals in which every element is a zerodivisor (recall that $x$ is a zerodivisor if there is $y \neq 0$ such that $xy = 0$.) Show that the set $\Sigma$ has maximal elements under inclusion and that every maximal element is a prime ideal. Conclude that the set of zerodivisors of $R$ is a union of prime ideals. (Remark: we will do better later and show that in a noetherian ring $R$ the set of zerodivisors is a union of finitely many primes called the associated primes of the ring.)

5. Let $R$ be a ring, and put the Zariski topology on $X = \text{Spec } R$. For any $f \in R$, define $X_f = V(f)^c = X \setminus V(f)$, i.e. the complement in $X$ of the closed set $V(f)$, the set of prime ideals containing $(f)$. The sets $X_f$ are open, and they are called principal open subsets of $X$. Prove the following statements.

(a) The sets $\{X_f | f \in R\}$ form a basis for the Zariski topology. In other words, every open set in $X$ is a union of such principal open subsets.

(b) $X_f \cap X_g = X_{fg}$.

(c) $X_f = \emptyset$ if and only if $f$ is nilpotent.

(d) $X_f = X$ if and only if $f$ is a unit.

(e) $X_f = X_g$ if and only if the ideals $(f)$ and $(g)$ have the same radical.

(f) $X$ is quasi-compact (this means that every open covering of $X$ has a finite subcover. This property is generally just called compact in topology). (Hint: show it is enough to consider a cover by a family of principal open sets $X_{f_\alpha}$. Show that the $f_\alpha$ generate the unit ideal, and so finitely many of them generate the unit ideal.)

6. Let $\phi : A \to B$ be a ring homomorphism. Let $X = \text{Spec } A$ and $Y = \text{Spec } B$ with their Zariski topologies. If $Q \in Y$, then $\phi^{-1}(Q) \in X$. This gives a map $\phi^* : Y \to X$ defined by $Q \mapsto \phi^{-1}(Q)$. Prove the following:
(a) If \( f \in A \) then \((\phi^*)^{-1}(X_f) = Y_{\phi(f)}\). Hence \( \phi^* \) is continuous.

(b) If \( \phi \) is surjective, then \( \phi^* \) is a homeomorphism of \( Y \) onto the closed subset \( V(\ker \phi) \) of \( X \) (with the subspace topology).

(c) If \( \phi \) is injective, then \( \phi^*(Y) \) is dense in \( X \).

(d) Let \( A \) be a ring with precisely two prime ideals \( 0 \subseteq m \) (for example, \( F[[x]] \) where \( F \) is a field is such a ring). In particular \( A \) is a domain and we let \( K \) be its field of fractions. Let \( B = (A/m) \times K \) and define \( \phi : A \to B \) by \( \phi(x) = (x + m, x) \). Show that \( \phi^* \) is bijective but not a homeomorphism.