Math 200c Spring 2015 Homework 2

Due 4/24/2015 in class or under Rob Won’s office door by 3pm

**Reading assignment**: Read Chapters 3-5 of Atiyah-Macdonald.

**Assigned problems (all to be turned in).**

1. Let $X$ be a topological space. Recall that a subset $Y$ of $X$ is then a topological space with the **subspace topology** if we declare the closed sets in $Y$ to be exactly those sets of the form $(Y \cap Z)$ where $Z$ is closed in $X$. Recall also that a map $f : X \to W$ of topological spaces is **continuous** if for every closed subset $Z$ of $W$, $f^{-1}(Z)$ is closed in $X$. A **homeomorphism** is a continuous map which is bijective and such that the inverse map is also continuous.

   Let $R$ be a ring and $S$ a multiplicative system in $R$. Consider the ring homomorphism $\phi : R \to S^{-1}R$ given by $r \mapsto r/1$, and the associated map of spectra $\phi^* : \text{Spec } S^{-1}R \to \text{Spec } R$ given by $\phi^*(Q) = \phi^{-1}(Q)$.

   (a). Let $Z$ be the image of $\phi^*$, with the topology it gets as a subspace of $\text{Spec } R$. Show that $\phi^*$ induces a homeomorphism from $\text{Spec } S^{-1}R$ to $Z$.

   (b). In case $S = R \setminus P$ for some prime ideal $P$, show that $R_P = \text{Spec } S^{-1}R$ is homeomorphic to the subset $Z$ of $\text{Spec } R$ consisting of all primes $Q$ contained in $P$. Show that $Z$ is also equal to the intersection of all open subsets $U$ of $\text{Spec } R$ such that $P \in U$. (This underlies the intuition that geometrically $\text{Spec } S^{-1}R$ is telling us about "local" behavior near the point $P$.)

2. (a). A ring $R$ is **reduced** if its nilradical is 0. Show that being reduced is a local property: $R$ is reduced if and only if the localization $R_P$ is reduced for all prime ideals $P$ of $R$.

   (b). Show that being a domain is not a local property. More specifically, show that if $R$ is a domain then $R_P$ is a domain for all prime ideals $P$ of $R$, but that the converse need not hold.

3. (a). Let $R$ be a ring and $S$ a multiplicative system. Show that if $J_1$ and $J_2$ are ideals of $S^{-1}R$ with $J_1 \subseteq J_2$, then $J_1^c \subseteq J_2^c$. Using this, prove that if $R$ is either noetherian or artinian, then the localized ring $RS^{-1}$ has the same property.
(b). Suppose that $f_1, \ldots, f_m \in R$ are elements which generate the unit ideal, i.e. such that $(f_1, \ldots, f_m) = R$. Prove that if $R_{f_i}$ is noetherian for all $1 \leq i \leq m$, then $R$ is noetherian. (As always, $R_f$ means the localization of $R$ at the multiplicative system $\{1, f, f^2, \ldots, \}$.)

4. (a). Let $S$ be a multiplicative system of a ring $R$. Suppose that $I$ is an ideal of $R$ such that $I \cap S = \emptyset$. Prove that the collection of all ideals $J$ such that $I \subseteq J$ and $J \cap S = \emptyset$ has a maximal element, and that any such maximal element is a prime ideal.

(b). Let $R$ be a nonzero ring and let $\Sigma$ be the set of all multiplicatively closed subsets $S$ of $R$ such that $0 \not\in S$. Show that every $S \in \Sigma$ is contained in a maximal element of $\Sigma$ and that $T \in \Sigma$ is maximal if and only if $T = R \setminus P$ for some minimal prime ideal $P$ of $R$.

(c). Let $R$ be a ring and let $D$ be the set of zerodivisors of $R$. Using part (b), prove that every minimal prime $P$ of $R$ is contained in $D$.

5. Many different multiplicative systems can lead to essentially the same localization, and this problem explores this phenomenon. Let $S \subseteq T$ be multiplicatively closed subsets of a ring $R$. Let $\phi : S^{-1}R \to T^{-1}R$ be defined by $a/s \mapsto a/s$; it is easy to see this is a ring homomorphism. Prove that the following are equivalent:

(a) $\phi$ is bijective.
(b) For each $t \in T$, $t/1$ is a unit in $S^{-1}R$.
(c) For each $t \in T$ there exists $x \in R$ such that $xt \in S$.
(d) Every prime ideal which meets $T$ also meets $S$.

(Hint: problem 4(a) can be used to prove that (d) implies (c).)