Note

On Induced Subgraphs of the Cube

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Consider the usual graph $Q^n$ defined by the $n$-dimensional cube (having $2^n$ vertices and $n2^{n-1}$ edges). We prove that if $G$ is an induced subgraph of $Q^n$ with more than $2^{n-1}$ vertices then the maximum degree in $G$ is at least $(\frac{1}{2} - o(1)) \log n$. On the other hand, we construct an example which shows that this is not true for maximum degree larger than $\sqrt{n} + 1$. © 1988 Academic Press, Inc.

1. Preliminaries

Denote by $Q^n$ the graph of the $n$-dimensional cube, i.e., the vertex set of $Q^n$ consists of all the $(0, 1)$-vectors of length $n$, and two vectors $x, y \in \{0, 1\}^n$ are adjacent if they differ from each other in exactly one place.

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component. For a graph $G = (V, E)$ we denote the maximum degree by $\Delta(G)$, i.e.,

$$\Delta(G) = \max_{v \in V(G)} \deg_G(v).$$

The average degree $\bar{d}(G)$ is defined to be $\sum_{v \in V(G)} \deg_G(v)/|V(G)|$. We say $G \in Q^n(N)$ if $G$ is an induced subgraph of $Q^n$ with $N$ vertices, i.e., $|V(G)| = N$, $V(G) \subseteq \{0, 1\}^n$, and $E(G) = E(Q^n) \cap (V(G) \times V(G))$.

$Q^n$ is a bipartite graph, so we have a $G \in Q^n(2^{n-1})$ without any edge, namely, $G^n_{\text{odd}}$ and $G^n_{\text{even}}$, where $V(G^n_{\text{odd}}) = \{x \in \{0, 1\}^n : \sum_{i=1}^n x_i \equiv 1 \pmod{2}\}$, $V(G^n_{\text{even}}) = \{0, 1\}^n - V(G^n_{\text{odd}})$. Our main result shows that even though the average degree of a graph $G \in Q^n(2^{n-1} + 1)$ can be very small (only $2n/(2^{n-1} + 1)$), these graphs must have large degree.

**Theorem 1.1.** Let $G$ be an induced subgraph of $Q^n$ with at least $2^{n-1} + 1$ vertices. Then for some vertex $v$ of $G$ we have

$$\deg_G(v) > \frac{1}{2} \log n - \frac{1}{2} \log \log n + \frac{1}{2}. \quad (1.1)$$

On the other hand, there exists a $G \in Q^n(2^{n-1} + 1)$ with

$$\Delta(G) < \sqrt{n} + 1. \quad (1.2)$$

2. Related Results and Problems from Computer Science

A (Boolean) function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ is said to depend on coordinate $i$ if there exists an input vector $x$ such that $f(x)$ differs from $f(x^{(i)})$, where $x^{(i)}$ agrees with $x$ in every coordinate except the $i$th. In this case $x$ is said to be critical for $f$ with respect to $i$. The function $f$ is called nondegenerate if it depends on all $n$ coordinates. For an input vector $x$, let $c(f, x)$ denote the number of coordinates $i$ such that $x$ is critical for $f$ with respect to $i$, and let $c(f) := \max\{c(f, x) : x \in \{0, 1\}^n\}$. $c(f)$ is called the critical complexity of $f$. This notion is due to Cook and Dwork [3] and Reischuk [5], who showed that $\log_A c(f)$ is a lower bound to the time needed by a parallel RAM to compute the function $f$ (where $A = \frac{1}{2}(5 + \sqrt{21}) = 4.7\ldots$). (A parallel RAM is a collection of synchronous parallel processors sharing a global memory with no write-conflicts allowed. For precise definitions, see [1].) Simon [6] showed that the critical complexity of any nondegenerate Boolean function is at least

$$\alpha(n) := \frac{1}{2} \log n - \frac{1}{2} \log \log n + \frac{1}{2}, \quad (2.1)$$
which implies a $O(\log \log n)$ lower bound for parallel complexity. More results on this topic can be found in [7].

Call a subgraph $G$ of $Q^n$ nondegenerate if $E(G)$ contains edges from each of the $n$ directions. Thus, the crucial point of the above problem can be reformulated as follows:

Let $U, V$ be a partition of $\{0, 1\}^n$ and consider the induced bipartite graph $G(U, V)$. If $G(U, V)$ is nondegenerate then $\Delta(G) > \alpha(n)$. (2.2)

This is completely analogous to our theorem (even the proof is similar). However, we need a slightly more powerful lemma (see Lemma 4.1). Reischuk (see [5]) has a simple example proving that in (2.2), $\Delta(G) = \lfloor \log n \rfloor + 2$ is possible, and it is very likely that this is the right value of

$$b(n) = \min \{\Delta(G); G \text{ as is in (2.2)}\}.$$

Another interesting property of the induced bipartite graphs is proved by Ben-Or and Linial [2] (also dealing with a problem arising in theoretical computer science):

If $U, V$ is a partition of $\{0, 1\}^n$ then there exists a direction $i$ such that at least $\min\{|U|, |V|\}/n$ edges go from $U$ to $V$ parallel to $i$. (2.3)

They have an upper bound of $\log n \min\{|U|, |V|\}/n$ and also this seems to be the right order of magnitude.

3. Proof of the Upper Bound

Denote the set of integers $\{1, 2, ..., n\}$ by $[n]$. Since there is a natural bijection between $\{0, 1\}^n$ and $2^{[n]}$, so we will speak about families of finite sets with the underlying set $[n]$. There exists a partition of $[n] = F_1 \cup \cdots \cup F_k$ such that $|k - \sqrt{n}| < 1$ and $|F_i| - \sqrt{n}| < 1$, $1 \leq i \leq k$. Define the family $X$ as follows: consider all the even sets (i.e., subsets of $[n]$ with cardinality an even number) which contain some $F_i$, $1 \leq i \leq k$, and all the odd sets which do not contain any $F_i$.

Claim 3.1. $|X| = 2^{n-1} + 1$ according to whether $n + k$ is odd or even.

Claim 3.2. For the subgraphs induced by $X$ and $2^{[n]} - X$ we have $\Delta \leq k$.

Remark. We can generalize the above construction in the following
way. Let $F \subseteq 2^{[n]}$ be a collection of finite sets. (Later we will see that it is enough to consider Sperner families, with $\bigcup F = [n]$.) Define

$$X(F) = \{ S \subseteq [n] : |S| \text{ is even, and there exists } F \in F \text{ with } F \subseteq S \} \cup \{ S \subseteq [n] : |S| \text{ is odd, } F \setminus S \neq \emptyset \text{ for all } F \in F \}.$$ 

Let $G(F)$ be the induced subgraph of $Q^n$ with vertex set $X(F)$, and $G'(F)$ the induced subgraph on $2^{[n]} - F$. The rank of $F$ is the largest size of its edges, i.e., $r(F) = \max \{|F| : F \in F\}$. Denote by $t(F)$ the maximum value of $t$ such that one can find $F_1, F_2, \ldots, F_t \in F$ and $x_i \in F_i$, $1 \leq i \leq t$, so that for $i \neq j$ we have $x_i \notin F_j$. In other words, $t(F)$ is the largest size of the disjointly representable subsystems of $F$.

**Proposition 3.3.** $\Delta(G(F)) \leq \max \{ r(F), t(F) \}$, and the same holds for $\Delta(G'(F))$.

**Proof.** If $(S, S')$ is an edge of $Q^n$, $S, S' \in X(F)$, and $S$ is even then $S' \subseteq S$. Moreover if $F \subseteq S$, $F \in F$, then $(S \setminus S') \in F$, so we have

$$\deg(S) \leq \left| \bigcap \{ F : F \in F, F \subseteq S \} \right| \leq r(F). \quad (3.1)$$

On the other hand, if $S$ is odd then $S \subseteq S'$ so there exists an $F \in F$, $F \subseteq S'$, $F \not\subseteq S$. Hence if $S \subseteq S'_1, S'_2, \ldots, S'_a$ then $F_1, \ldots, F_a$ (where $F_i \subseteq S'_i$) are disjointly representable, so $a \leq t(F)$. The statement $\Delta(G'(F)) \leq \max \{ r(F), t(F) \}$ can be proved in the same way. 

Now use the sieve method to determine the cardinality of $X(F)$. Let $F \subseteq [n]$, and $n \equiv \varepsilon \pmod{2}$ ($\varepsilon = 0$ or $1$). Then

$$\text{the number of even sets containing } F = \begin{cases} 2^{n-|F|} - 1 & \text{if } |F| < n, \\ 0 & \text{if } |F| = n, \text{ and } n \text{ is odd}, \\ 1 & \text{if } |F| = n, \text{ and } n \text{ is even}. \end{cases}$$

Similarly,

$$| \{ S : F \subseteq S \subseteq [n], |S| \text{ odd} \} | = \begin{cases} 2^{n-|F|} - 1 & \text{if } |F| < n, \\ \varepsilon & \text{if } |F| = n. \end{cases}$$

Let $F = \{ F_1, \ldots, F_N \}$. The cardinality of the first part of $X(F)$ is

$$\sum_{i \in [N]} (2^{n-|F_i|} - 1) = \sum_{i \in [N]} (2^{n-|F_i|} - 1) + \ldots, \quad (3.2)$$
where \((2^4)^*\) means \(2^A\) for \(A \geq 0\) and \(1 - \varepsilon\) for \(A = -1\). The cardinality of the second part of \(X(F)\) is
\[
2^{n-1} - \sum_{i \in [N]} (2^n - |F_i|-1)^{**} + \sum_{\{i,j\} \subset [N]} (2^n - |F_i \cup F_j|-1)^{**} - \cdots , \tag{3.3}
\]
where \((2^4)^{**}\) means \(2^A\) for \(A \geq 0\) and \(\varepsilon\) for \(A = -1\). We have \((2^4)^* - (2^4)^{**} = 0\) or \(1 - 2\varepsilon\) according to whether \(A \geq 0\) or \(A = -1\). So summing up (3.2) and (3.3) we have
\[
|X(F)| = 2^{n-1} + (1 - 2\varepsilon) \left[ \sum_{F_i \in F} 1 - \sum_{F_i, F_j \in F} 1 + \sum_{F_i, F_j, F_k \in F} 1 - \cdots \right]. \tag{3.4}
\]
Denote by \(f(F)\) the bracketed expression on the right-hand side of (3.4). It is clear that if \(F\) is a \(k\)-partition of \([n]\) (into nonempty parts) then \(f(F) = (-1)^{k+1}\), which implies
\[
|X(F)| = 2^{n-1} + (-1)^{n+k+1},
\]
proving Claim 3.4.

In general we are not able to calculate \(f(F)\) explicitly since it tends to get complicated. Some properties of \(f\) are:

(i) If \(F_0 = [n] \in F\) then \(f(F) = f(F - \{F_0\})\);

(ii) If \(F_0 = \emptyset \in F\) then \(f(F) = 0\);

(iii) If \(F = \{F_0, F_1, \ldots, F_N\}, \emptyset \neq F_0 \neq [n]\) then
\[
f(\{F_0, \ldots, F_N\} \mid [n]) = f(\{F_1, \ldots, F_N\} | [n])
- f(\{F_1 - F_0, \ldots, F_n - F_0\} \mid [n] - F_0);
\]

(iv) If \(F_0 \neq \emptyset\) and for some \(F_i \supset F_0\) then \(f(F) = f(F - \{F_0\})\).

**Proposition 3.4.** Suppose \(f(F) \neq 0\). Then \(\max\{r(F), t(F)\} \geq \sqrt{n}\).

**Proof.** Suppose that \(|F| < \sqrt{n}\) holds for all \(F \in F, f(F) \neq 0\) implies that \(|\cup F| = n\). Let \(\{F_1, \ldots, F_s\}\) be a minimal subfamily of \(F\) with \(\cup F = [n]\). Then \(\{F_1, \ldots, F_s\}\) is disjointly representable and \(s \geq \sqrt{n}\).

However, it may be possible that using a more complicated \(F\) with large \(f(F)\) and deleting some members of \(X(F)\) (but fewer than \(f(F)\)) one can obtain a \(G \in G^n(2^{n-1} + 1)\) with \(A(G) \ll \sqrt{n}\).
4. Proof of the Lower Bound

We begin with a lemma.

**Lemma 4.1.** Let \( G \) be a subgraph of the cube with average degree \( \bar{d} \). Then \( |V(G)| \geq 2^{d} \).

A similar lemma was used in [6], where \( |V(G)| \geq 2^{\min \deg(v)} \) was proved. We point out that a related result of Kleitman et al. [4] immediately implies Lemma 4.1 in the case that \( \bar{d} \) is an integer.

**Proof.** We use induction on \( |V(G)| \). Split \( Q^n \) into two \((n-1)\)-dimensional subcubes \( Q_1 \) and \( Q_2 \) such that \( V_1 = Q_1 \cap V(G) \neq \emptyset \) and \( V_2 = Q_2 \cap V(G) \neq \emptyset \). Suppose that \( |V_2| \geq |V_1| \) and there are \( s \) edges between \( V_1 \) and \( V_2 \) in \( G \) (so that \( |V_1| \geq s \)). The restriction of \( G \) to \( V_i \), \( i = 1, 2 \), is denoted by \( G_i \). The induction hypothesis gives

\[
|V_i| \log |V_i| \geq \sum_{v \in V_i} \deg_{G_i}(v) = \sum_{v \in V_i} \deg_G(v) - s,
\]

so that

\[
|V_1| \log |V_1| + |V_2| \log |V_2| + 2s \geq \sum_{v \in V(G)} \deg_G(v). \tag{4.1}
\]

However,

\[
(|V_1| + |V_2|) \log(|V_1| + |V_2|) \\
\geq |V_1| \log |V_1| + |V_2| \log |V_2| + 2 |V_1|
\]

if \( |V_2| \geq |V_1| \). (Here we used the fact that the base of the logarithm is 2.)

Of course, \( Q^n \) is decomposable into two \((n-1)\)-dimensional subcubes \( Q'_1, Q'_2 \), \( 1 \leq i \leq n \), in natural ways according to the \( n \) directions. We prove slightly more than (1.1).

**Lemma 4.2.** Suppose \( G \in Q^n(2^n-1) \) and \( G \) contains edges from all the \( n \) directions. Then \( \Delta(G) \geq \alpha(n) \).

This immediately implies (1.1). Indeed, let \( G \in Q^n(2^n-1+b) \) with \( \Delta(G) < (n-1)/2 \). Delete \( b \) vertices from \( G \) arbitrarily. In the resulting graph \( G_0 \) every direction must occur, since otherwise \( \Delta(G) \geq (n-1)/2 \) would be forced.

**Proof of Lemma 4.2.** Let \( X_i = \{ x \in V(G): x^{(i)} \in V(G) \} \), i.e., the set of
endpoints of the edges of $G$ in direction $i$. Define $Y_i = \{ y \notin V(G) : y^{(i)} \notin V(G) \}$, $A_i = V(Q^n) - X_i - Y_i$. Then

$$|X_i| = |Y_i| > 0.$$ 

Let $\Delta = \Delta(G)$ and consider a pair $x, x^{(i)} \in X_i$.

Claim 4.3. $x$ has at most $(2\Delta - 2)$ neighbours in $A_i$.

Proof. Let us denote the neighbours of $x$ in $A_i$ by $x^{(1)}, ..., x^{(s)}$. Then $x^{(1)}, ..., x^{(s)}$ are neighbours of $x^{(0)}$ in $A_i$ and either $x^{(h)}$ or $x^{(l)}$ belong to $V(G)$. Thus, $s \leq 2(\Delta - 1)$.

Claim 4.3 implies that every $x \in X_i$ has at least $(n - 2\Delta + 1)$ neighbours in $Y_i$. Hence

$$|E(G(X_i \cup Y_i))| \geq \frac{1}{2} |X_i| + \frac{1}{2} |Y_i| + (n - 2\Delta + 1) |X_i|,$$

implying

$$d(G(X_i \cup Y_i)) \geq n - 2\Delta + 2.$$ 

Lemma 4.1 gives

$$|X_i| \geq 2^{n-2\Delta+1}.$$  \hspace{1cm} (4.2)

Counting the degrees in $V(G)$ we have

$$\Delta \cdot 2^{n-1} \geq \sum_{v \in V(G)} \deg_G(v) = \sum_{i=1}^{n} |X_i| \geq n2^{n-2\Delta+1}.$$ 

An easy calculation now gives $\Delta \geq a(n)$, as desired.

References

