ON THE RAMSEY NUMBERS $N(3, 3, ..., 3; 2)$

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Abstract. The main results of this paper are $N(3, 3, 3, 3; 2) > 50$ and $f(k+1) \geq 3f(k) + f(k-2)$, where $f(k) = N(3, 3, ..., 3; 2) - 1$ for $k \geq 3$.

1. Introduction

The theorem of Ramsey says: Given integers $S_1, S_2, S_3, ..., S_k$, where $S_1, S_2, ..., S_k \geq 2$, there exists a minimum integer $N(S_1, S_2, ..., S_k; 2)$ such that the following property is valid for all $n \geq N(S_1, S_2, ..., S_k; 2)$. Let the edges of a complete graph of $n$ vertices be colored in $k$ colors, then there exists a subset of $S_i$ vertices with all its interconnecting segments of the $i^{th}$ color for some $i \leq k$.

Now, consider the case of $S_1 = S_2 = ... = S_k = 3$. Let

$$f(k) = N(3, 3, ..., 3; 2) - 1.$$  

The problem reduces to the following: If the edges of $K_n$ are colored in $k$ colors and if $n > f(k)$, then there exists some triangle with all its sides in the same color. Find $f(k)$.

It is known [1] that $2^k \leq f(k) \leq [k! e]$. Particularly, $f(1) = 2, f(2) = 5, f(3) = 16$. Whitehead [3, 4] has proved $f(4) \geq 49$. It will be shown here that $f(k+1) \geq 3f(k) + f(k-2)$ for $k \geq 3$ and, in particular, $f(4) \geq 50$, thus $N(3, 3, 3, 3; 2) > 50$.

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2. \( N(3, 3, 3, 3; 2) > 50 \)

Consider the symmetric \(16 \times 16\) matrix:

\[
T_3(x_0, x_1, x_2, x_3) =
\begin{align*}
&x_0 \\
&x_1 x_0 \\
&x_1 x_2 x_0 \\
&x_1 x_2 x_3 x_0 \\
&x_1 x_2 x_3 x_2 x_0 \\
&x_1 x_3 x_3 x_2 x_0 \\
&x_1 x_3 x_2 x_3 x_2 x_0 \\
&x_2 x_3 x_2 x_2 x_1 x_1 x_0 \\
&x_2 x_2 x_3 x_1 x_1 x_2 x_3 x_0 \\
&x_2 x_2 x_1 x_3 x_2 x_1 x_3 x_1 x_0 \\
&x_2 x_1 x_2 x_3 x_2 x_1 x_3 x_1 x_0 \\
&x_2 x_1 x_2 x_1 x_3 x_3 x_2 x_0 \\
&x_3 x_2 x_1 x_1 x_3 x_3 x_3 x_2 x_0 \\
&x_3 x_1 x_2 x_3 x_3 x_1 x_3 x_2 x_3 x_1 x_0 \\
&x_3 x_1 x_3 x_2 x_1 x_3 x_3 x_2 x_1 x_2 x_0 \\
&x_3 x_3 x_3 x_1 x_2 x_3 x_1 x_2 x_1 x_0 \\
&x_3 x_3 x_1 x_3 x_3 x_2 x_3 x_1 x_2 x_1 x_0
\end{align*}
\]

It is known \([2]\) that \(T_3(0,1,2,3)\) is the incidence matrix of one of the two non-isomorphic edge-coloring schemes of \(K_{16}\) without any one-color triangles.

Now construct the \(50 \times 50\) incidence matrix in the following way:

\[
T_4(0,1,2,3,4) =
\begin{array}{c|c|c|c|c|c}
A & & & & & \\
\hline
& & & & & \\
D & B & & & & \\
\hline
& & & & & \\
E & F & C & & & \\
\hline
11 & 22 & 33 & 3 & 0 & \\
11 & 22 & 33 & 3 & 4 & 0
\end{array}
\]

\[1\) Dr. G.J. Porter proved 2 independently in Univ. of Pennsylvania.\]
where \( A = T_3(0, 2, 3, 4) \),
\[ B = T_3(0, 3, 1, 4) \],
\[ C = T_3(0, 1, 2, 4) \],
\[ D = T_3(3, 2, 1, 4) \],
\[ E = T_3(2, 1, 3, 4) \],
\[ F = T_3(1, 3, 2, 4) \].

If there are some one-color triangles with vertices \( i, j, k \), then \( t_{i,j} = t_{k,j} = t_{k,i} \). We may assume \( k > i > j \) without loss of generality.

Case 1: \( t_{i,j} = t_{k,j} = t_{k,i} = 4 \).

We notice that \( t_{m,n} = t_{m',n'} = 4 \) if \( m \equiv m' \) (mod 16), \( n \equiv n' \) (mod 16) for \( m, m', n, n' \leq 48 \). Hence we may pick \( i', j', k' \) such that \( i \equiv i', j \equiv j', k \equiv k' \) (mod 16) and \( i', j', k' \leq 16 \); then \( t_{i',j} = t_{k',j} = t_{k',i} = 4 \). This contradicts the fact that \( T_3 \) is the incidence matrix of a coloring without a one-color triangle. In case of \( k = 50, i = 49 \), we know that \( t_{50,49} = 4 \) and that \( t_{j,49}, t_{j,50} \) do not have value 4 for any \( j \neq 49, 50 \).

Case 2: \( t_{i,j} = t_{k,j} = t_{k,i} = 2 \).

(1) \( 16 \geq j \geq 1, 16 \geq i \geq 1, t_{i,j} \) is in part \( A \).

(a) If \( t_{k,j} \) is in part \( A \), then \( t_{k,i} \) is in part \( A \). This contradicts the structure of \( T_3 \).

(b) If \( t_{k,j} \) is in part \( D \), then \( t_{k,i} \) is in part \( D \). We know that \( t_{i+16,j} = t_{i,j} = 2 \). Then \( t_{i+16,j} = t_{k,j} = t_{k,i} = 2 \). Impossible.

(c) If \( t_{k,j} \) is in part \( E \), then \( t_{k,i} \) is in part \( E \). But there is only one entry with value 2 in each row of \( E \). Contradiction.

(2) \( 16 \geq j \geq 1, 32 \geq i \geq 17, t_{i,j} \) is in part \( D \).

(a) If \( t_{k,j} \) is in part \( D \), then \( t_{k,i} \) is in part \( B \). But there is no entry with value 2 in \( B \). This is impossible.

(b) If \( t_{k,j} \) is in part \( E \), then \( t_{k,i} \) is in part \( F \). It is known that only the entries on the diagonal are of value 2 in \( E \). Hence \( k = 32+j \).

We have \( t_{i,j} = t_{32+j,i} = t_{32+j,i} = 2 \). But \( t_{32+j,i} = 3 \) if \( t_{k,j} = 2 \). Contradiction.

(3) \( 16 \geq j \geq 1, 50 \geq i \geq 33, t_{i,j} \) is in part \( E \). There is only one entry with value 2 in part \( E \). This is impossible.

(4) \( 32 \geq j \geq 17, 32 \geq i \geq 17, t_{i,j} \) is in part \( B \). This is impossible because there is no entry with value 2 in \( B \).

(5) \( 32 \geq j \geq 17, 48 \geq i \geq 33, t_{i,j} \) is in part \( F \).

(a) \( t_{k,j} \) is in part \( F \) and \( t_{k,i} \) is in part \( C \) and \( t_{k,i} = t_{k,i-16} = 2 \). Then \( t_{i,j}, t_{k,j}, t_{k,i-16} \) are all in \( F \) and all with value 2. This contradicts the structure of \( T_3 \).
(b) $k = 49$ or $50$. In this case, $t_{k,i} = 3 \neq t_{i,j}$.

(6) $i = 49$, $32 \geq j \geq 17$, $k = 50$. Then $t_{50,49} = 4 \neq 2$. Impossible.

(7) $48 \geq j \geq 33$, $48 \geq i \geq 33$, $t_{i,j}$ is in part $C$. $t_{k,j}$, $t_{k,i}$ is in part $C$.

This contradicts the structure of $T_3$.

Case 3: $t_{i,j} = t_{k,j} = t_{k,i} = 1$. This is impossible. The proof is similar to case 2.

Case 4: $t_{i,j} = t_{k,j} = t_{k,i} = 3$. Similarly impossible.

Hence we prove that $T_4(0,1,2,3,4)$ is the incidence matrix of the coloring of $K_{50}$ without a one-color triangle.

Thus, $f(4) \geq 50$, i.e., $N(3,3,3,3;2) > 50$.

3. $f(k + 1) \geq 3f(k) + f(k - 2)$

The result in Section 2 can be generalized to any $k \geq 4$.

Let $T_k(x_0, x_1, ..., x_k)$ be the incidence matrix of the coloring of the complete graph of $n_k$ vertices without a one-color triangle in $k$ colors.

Similarly, we construct $T_{k+1}(0,1,2, ..., k+1)$ as shown in Diagram 1.

<table>
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<td>1 ....</td>
<td>22 ...</td>
<td>33 ...</td>
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</tr>
</tbody>
</table>

Diagram 1.

$A = T_k(0, 2, 3, 4, 5, ..., k+1)$, $B = T_k(0, 3, 1, 4, 5, ..., k+1)$,
$C = T_k(0, 1, 2, 4, 5, ..., k+1)$, $D = T_k(3, 2, 1, 4, 5, ..., k+1)$,
$E = T_k(2, 1, 3, 4, 5, ..., k+1)$, $F = T_k(1, 3, 2, 4, 5, ..., k+1)$,
$G = T_{k-2}(0, 4, 5, ..., k+1)$.

The proof that such a coloring has no one-color triangle is quite similar to the proof in Section 2. Hence we have $f(k+1) \geq 3f(k) + f(k - 2)$.
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References