ON THE COVERINGS OF GRAPHS

F.R.K. Chung

Bell Laboratories, 600 Mountain Avenue, Murray Hill, NJ 07974, USA

Received 21 September 1978
Revised 12 October 1979

Let \( \rho(n) \) denote the smallest integer with the property that any graph with \( n \) vertices can be covered by \( \rho(n) \) complete bipartite subgraphs. We prove a conjecture of J.-C. Bermond by showing \( \rho(n) = n + o(n^{1/14+\epsilon}) \) for any positive \( \epsilon \).

1. Introduction

Suppose \( G \) is a connected graph\(^1\) with vertex set \( V(G) \) and edge set \( E(G) \). A covering of \( G \) is a family of subgraphs, say \( G_1, G_2, \ldots, G_t \), having the property that each edge of \( G \) is contained in at least one graph \( G_i \), for some \( i \). If all \( G_i \), \( 1 \leq i \leq t \), belong to a specified class of graphs \( H \), such a covering is called an \( H \)-covering of \( G \). If we require all subgraphs in the covering to be edge-disjoint, the covering is also called a decomposition of \( G \).

One of the fundamental topics in graph theory is to study the coverings and the decompositions of graphs. Much work has been done on \( H \)-covering and \( H \)-decompositions for various classes \( H \) (see [3]). In this note, we prove a conjecture of J.-C. Bermond [1] on \( B \)-coverings of graphs, where \( B \) is the set of complete bipartite graphs, as follows:

Let \( \rho(n) \) be the smallest number with the property that any graph on \( n \) vertices has a \( B \)-covering consisting of no more than \( \rho(n) \) subgraphs. It was conjectured by J.-C. Bermond that

\[
\lim_{n \to \infty} \frac{\rho(n)}{n} = 1
\]

We will show that this conjecture is true.

2. Preliminaries

In the remaining part of the paper, a covering usually means a \( B \)-covering. We note that the complete graph \( K_n \) has a covering of \( \lceil \log_2 n \rceil \) complete bipartite

\(^1\) We only consider graphs without loops or multiple edges. The reader is referred to [9] for undefined terminology.
graphs, where \([x]\) denotes the least integer greater than or equal to \(x\). A path on \(n\) vertices has a covering of \([\frac{1}{2}n]\) complete bipartite subgraphs where \([x]\) denotes the greatest integer less than or equal to \(x\). It is easy to see that \(\lim_{n \to \infty} \rho(n)/n \geq \frac{2}{3}\) by considering the graph which is the vertex-disjoint union of \([\frac{1}{3}n]\) copies of \(K_3\). It was first suspected that \(\frac{2}{3}\) might be the value of \(\lim_{n \to \infty} \rho(n)/n\). However, \(\rho(n)\) can be shown to be much greater than \(\frac{2}{3}n\) for large \(n\). In fact, \(\rho(n)\) is fairly close to its upper bound \(n - 1\). We note that a graph on \(n\) vertices can be covered by \(n - 1\) stars i.e., complete bipartite graphs \(K_{1,n}\). Therefore we have

\[
\rho(n)/n < 1.
\]

In the next section, we will show

\[
\lim_{n \to \infty} \rho(n)/n = 1
\]

by proving a lower bound \(n - n^{11/14+\epsilon}\) for any \(\epsilon > 0\).

3. A lower bound

We will show the following.

**Main Theorem.** (i) For infinitely many \(n\), we have \(\rho(n) > n - n^{3}\).

(ii) For any positive \(\epsilon\), we have

\[
\rho(n) > n - n^{11/14+\epsilon}
\]

for sufficiently large \(n\).

**Proof.** Let us consider a graph \(G\) on \(n = q^2 + q + 1\) vertices, where \(q\) is a prime power.

It is well known [11] that there exists a difference set \(\{d_1, \ldots, d_{q+1}\} \subset \{1, \ldots, q^2 + q + 1\}\) such that for any \(x = 0 \pmod{q^2 + q + 1}\), there is exactly one ordered pair \((d_i, d_j)\), such that \(d_i - d_j = x \pmod{q^2 + q + 1}\). Now in the graph \(G\), \(v_i\) is adjacent to \(v_j\) if and only if \(i \neq j\) and \(i + j = d_k \pmod{q^2 + q + 1}\) for some \(k\).

Suppose there is a four-cycle on distinct vertices \(v_s, v_t, v_x, v_y\). Then clearly we have

\[
x + y = d_i, \quad y + z = d_j.
\]

where all congruences are to modulus \(q^2 + q + 1\). Therefore, \(x - z = d_i - d_j\).

Similarly we have

\[
x + w = d_{i'}, \quad z + w = d_{j'}.
\]

where

\[
i' \neq i'', \quad j' \neq j''.
\]

Then \(d_i - d_j = d_{i'} - d_{j'}\).
This contradicts the definition of a difference set. We conclude that this graph \( G \) does not contain any four-cycle as a subgraph. Thus any complete bipartite subgraph of \( G \) must be a star. We note that this graph \( G \) has also been used in [2, 4, 8].

Now, we consider a covering of \( G \) consisting of stars \( S_1, \ldots, S_s \). Let \( u_i, 1 \leq i \leq s \), be a vertex of \( S_i \) such that \( u_i \) is adjacent to every other vertex in \( S_i \).

The following observations are immediate.

**Fact 1.** Let \( v \) be a vertex in \( V(G) - U \) where \( U = \{u_i : 1 \leq i \leq s\} \). Any vertex adjacent to \( v \) must belong to \( U \).

**Fact 2.** We define \( N(v) = \{u \in U : \{u, v\} \in E(G)\} \). Then \( N(v) \) has \( q \) or \( q + 1 \) elements and the union of \( N(v) \) over all \( v \in V(G) - U \) is \( U \).

**Fact 3.** For \( u, v \in P, u \neq v \), we have

\[ |N(u) \cap N(v)| \leq 1. \]

We now consider a \((0, 1)\)-matrix \( A = \{A_{ij} : 1 \leq i \leq t = n - s, 1 \leq j \leq s\} \) defined by

\[ A_{ij} = \begin{cases} 1 & \text{if } u_i \supseteq N(p_j), \\ 0 & \text{otherwise} \end{cases} \]

where \( V(G) - U = \{p_1, p_2, \ldots, p_t\} \).

We note that every row of \( A \) has sum \( q \) or \( q + 1 \). We evaluate in two ways the sum of the inner products of the rows:

\[ \sum_{i=1}^{t} \sum_{j=1}^{s} \sum_{k \neq i}^{s} A_{ik}A_{jk} \leq t(t - 1). \tag{1} \]

We note that (1) follows from Fact 3 and the left-hand side of (1) is equal to

\[ \sum_{k=1}^{s} \sum_{i=1}^{t} \sum_{j=1}^{t} A_{ik}A_{jk}. \tag{2} \]

Let \( q_i \) denote the column sum of the \( i \)-th column, \( 1 \leq i \leq s \). Then (2) is equal to

\[ \sum_{k=1}^{s} q_k(q_k - 1). \]

Note that

\[ t(q + 1) = \sum_{k=1}^{s} q_k \geq tq. \]

Therefore

\[ \sum_{k=1}^{s} q_k^2 \geq t^2q^2/s. \]

From (1) we obtain

\[ t(t - 1) \geq t^2q^2/s - t(q + 1). \]
By substituting \( s = n - t = q^2 + q + 1 - t \), we have

\[
t^2 - t - q(q^2 + q - 1) \leq 0.
\]

Thus

\[
t < \sqrt{q(q^2 + q + 1)} + \frac{3}{4} + \frac{1}{4} < n^3 \quad \text{and} \quad s > n - n^3.
\]

Therefore, we have shown that for \( n = q^2 + q + 1 \), \( q \) a prime power, we have

\[
\rho(n) \geq n - n^3.
\]

Now suppose \( n \) is an arbitrary integer. It is recently shown [10] that there exists a prime \( q \) between \( \sqrt{n-1} \) and \( \sqrt{n-n^{3+\sigma}} \) for any given \( \sigma > 0 \) for large enough \( n \). It is easy to see that \( n \approx q^2 + q + 1 \). We also note that \( \rho(n) \approx \rho(n') \) for any \( n' \) with \( n \approx n' \) since any graph on \( n' \) vertices can be viewed as a graph on \( n \) vertices.

Therefore we have

\[
\rho(n) \geq \rho(q^2 + q + 1) \\
\geq (q^2 + q + 1) - (q^2 + q + 1)^2 \geq n - n^{11/14 + \epsilon}
\]

for any given \( \epsilon > 0 \).

Thus, the main theorem is proved.

Professor Erdös [6] pointed out that a graph \( G \) on \( n \) vertices has either an independent set of size \( (c \log n) \) or it contains a complete subgraph on at least \( c \log n \) vertices for \( c = \log \frac{1}{2} n \log 2 \) since the Ramsey number

\[
r(a, b) < \left( \frac{a + b - 2}{a - 1} \right).
\]

In either case, \( G \) can always be covered by \( n - c' \log n \) complete bipartite subgraphs for some constant \( c' \).

**Concluding remarks**

The preceding results suggest a number of related problems, several of which we now mention:

1. Consider \( \rho'(n) = n - \rho(n) \). We know that \( \rho'(n) \) is between \( c_1 \log n \) and \( c_2 n^{11/14 + \epsilon} \) for any \( \epsilon > 0 \) and some constants \( c_1, c_2 \). What is the asymptotic behavior of \( \rho' \)?

2. For a given graph \( G \) and a specified family of graphs \( \mathcal{H} \), we define \( \rho(G; \mathcal{H}) \) to be the minimum number of subgraphs from \( \mathcal{H} \) needed to cover \( G \). We also define \( \rho(n, \mathcal{H}) \) to be the maximum value of \( \rho(G; \mathcal{H}) \) over all graphs \( G \) with \( n \) vertices.

Let \( \mathcal{P} \) denote the set of all simple paths. We can then ask whether any graph with \( n \) vertices can always be covered by \( \lceil \frac{1}{2} n \rceil \) simple paths, i.e., is the following true?
Conjecture. $\rho(n, P) = \lfloor \frac{1}{3} n \rfloor$.

We note that this is an analogue of the Gallai conjecture on the decomposition of graphs.

(3) Let $C$ denote the set of all simple cycles. We let $\rho(G; C) = 0$ if $G$ has a vertex of odd degree. It seems reasonable to conjecture that any graph with $n$ vertices can be covered by $\lfloor \frac{1}{3} n \rfloor$ simple cycles, i.e.,

Conjecture. $\rho(n, C) = \lfloor \frac{1}{3} n \rfloor$.

(We note that this a weaker version of the Hajós conjecture on the decomposition of graphs.)

(4) We can ask the question of determining $\rho(n, H)$ for $H$ being a class of graphs with certain specified properties, e.g., each graph has diameter $\leq x$, has chromatic number $\leq y$, has connectivity $\leq z$, etc.

We remark that V. Chvátal [5] has also proved the conjecture $\lim_{n \to \infty} \rho(n)/n = 1$ by showing $\rho(n) \geq n - n^{\frac{1}{2} + \epsilon}$, based on a probabilistic result of P. Erdős [7].

References