ON TRIANGULAR AND CYCLIC RAMSEY NUMBERS WITH k COLORS

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ABSTRACT

Define \( r(G;k) \) to be the smallest integer with the following property: For any \( n \geq r(G;k) \), color the edges of \( K_n \) in \( k \) colors, then there exists a monochromatic graph isomorphic to \( G \). In this paper, we discussed the bounds for \( r(K_3;k) \) and \( r(C_4;k) \).
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Let $G$ be a finite graph and $k$ be a positive integer. Define $r(G;k)$ to be the smallest integer with the following property: For any $n \geq r(G;k)$, color the edges of $K_n$ in $k$ colors; then there exists a monochromatic graph isomorphic to $G$. The existence of $r(G;k)$ is assured by Ramsey's theorem [1,2].

In the case of $G = K_3$ and $k = 2$, $r(K_3;2) = 6$. This is one of the most interesting fundamental problems that appeared in Putnam Mathematics Competition [3] in 1953. The problem can be stated as follows: Color the edges of $K_6$ in red or blue; then either a red triangle or a blue triangle exists.

In 1955 Greenwood and Gleason [4] proved that $r(K_3;3) = 17$ and $r(K_3;4) > 41$. The value of $r(K_3;4)$ is still unknown. Whitehead and Taylor [5] proved that $r(K_3;4) > 49$ in 1971. In 1972, G. J. Porter (unpublished) and the author [6] proved independently that $r(K_3;4) > 50$ and a lower bound for $r(K_3;k)$ was obtained. A simpler proof will now be presented.

**Theorem 1.** Let $f(k) = r(K_3;k) - 1$ and let $t = 0.103 \ldots$ be the only positive root of $x^3 + 6x^2 + 9x - 1 = 0$ and $C = 50t^2 = 0.5454 \ldots$. Then $f(k+1) \geq 3f(k) + f(k-2)$ for $k > 3$ and $f(k) \geq (3t)^k C$.

We need the following lemmas.

**Lemma 1:** The edges of $K_n$ can be colored in $k$ colors without any monochromatic triangle if and only if its adjacency matrix $A_n$ is the sum of $k$ symmetric binary matrices $M_1, M_2, \ldots, M_k$ where the componentwise product of $M_i$ and $M_i^2$ is zero for $i = 1, 2, \ldots, k$, i.e., $A_n = M_1 + M_2 + \ldots + M_k$ and $(M_iM_i^2)_{ij} = (M_i^2)_{ij} = 0$ for $i = 1, 2, \ldots, k$.

**Proof:** If the edges of $K_n$ are colored in $k$ colors without monochromatic triangles, then define

$$ (M_i)_{jm} = \begin{cases} 1 & \text{if the edge } (jm) \text{ has color } i \\ 0 & \text{otherwise} \end{cases} $$
Obviously \((M_1^2)_{1m}\) is the number of paths of length 2 joining points \(j\) and \(m\). But \((M_1^1)_{1m}\) should be zero when \((M_1^2)\) is non-zero. Hence \(A_n = M_1 + M_2 + \ldots + M_k\) and \(M_i^1 M_1^2 = 0\) for all \(i\).

Conversely, given \(k\) symmetric binary matrices \(M_1, M_2, \ldots, M_k\) with their sum \(A_n\) and \(M_i^1 M_i^2 = 0\) for \(i = 1, 2, \ldots, k\), we have a \(k\)-coloring of \(K_n\) without any monochromatic triangles.

**Proof of theorem.** The edges of \(K_f(k)\) can be colored in \(k\) colors without monochromatic triangles. By Lemma 1 there exist \(M_1, M_2, \ldots, M_k\) and \(N_1, N_2, \ldots, N_{k-2}\) such that

\[
A_f(k) = M_1 + M_2 + \ldots + M_k \quad \text{and} \quad M_i^1 M_1^2 = 0 \quad \text{for} \quad i = 1, 2, \ldots, k
\]

\[
A_f(k-2) = N_1 + N_2 + \ldots + N_{k-2} \quad \text{and} \quad N_j^1 N_j^2 = 0 \quad \text{for} \quad j = 1, 2, \ldots, k-2
\]

and let \(J\) be the \((k-2) \times f(k-2)\) matrix with all entries 1.

Let \(L_1, L_2, \ldots, L_{k+1}\) be square matrices of order \((3f(k) + f(k-2))\); then symmetric matrices are defined as follows:

\[
L_1 = \begin{pmatrix}
0 & M_2 & M_1 \\
M_2 & 0 & 0 \\
M_1 & 0 & 0
\end{pmatrix}
\]

\[
L_2 = \begin{pmatrix}
M_1 & 0 & 0 \\
0 & M_2 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

\[
L_3 = \begin{pmatrix}
M_2 & 0 & 0 \\
0 & M_1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

\[
L_i = \begin{pmatrix}
M_i & 0 & 0 \\
0 & M_i & 0 \\
0 & 0 & 0
\end{pmatrix} \quad \text{for} \quad i = 4, 5, \ldots, k+1.
\]

It is clear that \(L_1 + L_2 + \ldots + L_{k+1} = A_{3f(k) + f(k-2)}\) and \(L_i^1 L_i^2 = 0\) for \(i = 1, 2, \ldots, k+1\).
Since the complete graph $K_{3f(k) + f(k-2)}$ can be colored in $k+1$ colors without any monochromatic triangle,
\[ f(k+1) \geq 3f(k) + f(k-2) \quad \text{for} \quad k \geq 3. \]

From the above inequality we can get $f(k) \geq (3+t)^{k}C$ where $t = 0.103...$ is the only positive root of $x^3 + 6x^2 + 9x - 1 = 0$ and $C = 50t^2 = 0.3454...$

The classical upper bound [4] for $r(K_{i};k)$ is $[k/2] + 1$. Whitehead [5] proved $r(K_{j};4) \leq 65 [41e] + 1$. Combining these, we get the next inequality.

**Theorem 2.** $r(K_{j};k) \leq [k/(e-1/24)] + 1$.

From Theorems 1 and 2, we know that the limit of $k$'th root of $f(k)$ will be between $3+t$ and $e$ if it exists.

**Lemma 2.** $f(2k) \geq (f(k))^3$.

**Proof.** Let $f(k) = n$ so that the edges of $K_n$ can be $k$-colored without any monochromatic triangles.

Define $K_{n,j}$ with vertices the vectors $(i_1, i_2, \ldots, i_j)$, $i_s = 1, 2, \ldots, n$. Let $c_s$, $s = 1, \ldots, j, m = 1, \ldots, k$, be the $jk$ colors available.

The edge joining $(i_1, i_2, \ldots, i_j)$ and $(i'_1, i'_2, \ldots, i'_j)$ is colored in the color $c_{j,m}$ if and only if $i_1 = i'_1, \ldots, i_{j-1} = i'_{j-1}, i_j \neq i'_j$ and the edge joining $i'_j$ has color $m$.

It is clear that this gives a coloring of edges of $K_{n,j}$ without any monochromatic triangle in $k$ colors.

Therefore
\[ f(2k) \geq (f(k))^3. \]

**Theorem 3.** $\lim_{k \to \infty} (f(k))^{1/k}$ exists.

**Proof:** Let $x = \lim \sup (f(k))^{1/k}$

There exists an integer $m$ such that $f(m)^{1/m} > x-\epsilon$

For any $n \geq m/\epsilon$, $f(n)^{1/n} \geq f(m)^{[n/m]/n}$
\[ \geq f(m)^{(n/m)/n} \geq (x-\epsilon)^{(1-\epsilon)/\epsilon} \]

Hence $\lim \inf f(k)^{1/k} = \lim \sup f(k)^{1/k} \geq 3.103...$
Theorem 4. Let \( r(K_m;k) \) be the classical Ramsey number \( N(m,m,\ldots, m;2) \). Then
\[
\lim_{k \to \infty} r(K_m;k)^{1/k} \text{ exists for any } m \text{ and is greater than } m-1.
\]

Proof: By a similar method we can prove \( r(K_m;k)^{1/k} \leq r(K_m;k)^{1/k} \) and the limit exists.

Let \( \xi_m = \lim r(K_m;k)^{1/k} \). Then \( \xi_3 = 3.103 \ldots \). It is not known that \( \xi_3 \) is finite or infinite. It was shown in [7] that \( \xi_4 \geq 1.77, \xi_5 \geq 1.49, \xi_6 \geq 1.37, \xi_7 \geq 1.09, \xi_8 \geq 1.01 \), \( \xi_9 \geq 1.00 \) and \( \xi_m \) is strictly increasing.

Some upper and lower bounds for \( r(C_4,k) \) have been obtained.

Lemma 3. The edges of \( K_n \) can be colored in \( k \) colors without any monochromatic triangle if and only if the matrix \( A_n \) is the sum of \( k \) symmetric binary matrices \( H_1, H_2, \ldots, H_k \) where \( (H_i^2)_{jm} \leq 1 \) for \( j \neq m \) \( 1 \leq i \leq k \).

The proof of Lemma 3 is clear.

Lemma 4. Let \( H \) be an \( n \times n \) symmetric binary matrix and \( (H^2)_{jm} \leq 1 \) for \( j \neq m \). Then
\[
S = \sum_{i=1}^{n} \sum_{j=1}^{n} H_{ij} \leq n^2/n-3/4 + n/2.
\]

Proof: \( \sum_{j=1}^{n} \sum_{i=1}^{n} H_{ij} \leq 1 \) \( (i \neq k) \)

Sum over \( k = 1, \ldots, n, k \neq j \), to get
\[
\sum_{j=1}^{n} \sum_{k=1}^{n} H_{jk} \leq n-1,
\]
or
\[
\sum_{j=1}^{n} (r(i)-H_{jj}) \leq n-1,
\]
where \( r(i) \) is the \( i \)th column sum or row sum.

Then sum over \( j \), to get
\[
\sum_{j=1}^{n} r(i)^2 \leq \sum_{i=1}^{n} (H_{ij})^2 \leq n(n-1),
\]
\[
\sum_{i=1}^{n} r(i)^2 \leq n(n-1) + S.
\]

For any positive numbers \( r(1), r(2), \ldots, r(n) \),
\[
\sum_{i=1}^{n} r(i)^2 \geq \left( \sum_{i=1}^{n} r(i) \right)^2/n.
\]
Theorem 5. \( k^2 + k + 1 \geq r(C_n;k) \).

**Proof.** Let \( r(C_n;k) - 1 = n \). By Lemma 3 we know that
\[
A_n = \sum_{i=1}^{n} M_i
\]
and
\[
(M_i)_{jm} \leq 1 \text{ for } j \neq m, i = 1,2,\ldots, k.\]
There is some \( M_i \) with the property that
\[
\sum_{j=m}^{n} (M_i)_{jm} \geq n(n-1)/k.
\]
By Lemma 4, we have
\[
n/2 + n\sqrt{n-3/4} \geq n(n-1)/k.
\]
Then \( k^2 + k + 1 \geq n \).

The equality holds when the row sums of \( M_i \) are all equal to \( k+1 \). In this case \( M_i \) is the adjacency matrix of a projective plane. But there does not exist [8] an adjacency matrix of a projective plane of trace 0.

Hence
\[
k^2 + k + 1 > n
\]
and
\[
k^2 + k + 1 \geq r(C_n;k).
\]

Theorem 6. \( r(C_n;k) \geq k^2/16 \) for infinitely many \( k \)'s.

The proof is established by an explicit construction.

After the conference the author proved that \( r(C_n;k+1) \geq k^2 \) for any small \( \varepsilon \) and large \( k \) and that \( r(C_n;k) \) is asymptotically equal to \( k^2 \).
REFERENCES


