Proof. We will construct a $d$-partite graph $\tilde{H}$ as follows:

(i) $\tilde{H}$ has vertex set $A_1 \cup A_2 \cup \cdots \cup A_{d+1}$ with $|A_i| = 2n/d$ for each $i$;
(ii) For each $i$, no $u, v \in A_i$ are adjacent;
(iii) The edges between $A_i$ and $A_1 \cup A_2 \cup \cdots \cup A_{i-1}$ form a graph described in Claim 2.

It can be easily seen that $\tilde{H}$ has at most $4n^2 \log \log n / \log n$ edges. It suffices to prove that any graph $G$ with degree $d$ can be embedded in $\tilde{H}$. A nice result of Hajnal and Szemerédi [6] states that any graph with degree at most $d$ can be colored by $d+1$ colors in such a way that the sizes of the color classes differ by at most 1. Suppose $G$ has color classes $C_1, \ldots, C_{d+1}$. We will then embed $C_1$ into $A_1$, $C_2$ into $A_2$, and so on, as guaranteed by Claim 2.

Claim 4. There exists a graph $F(n)$ with $Cn^2 \log \log n \log n$ edges which contains all graphs on $n$ edges where $C$ is an absolute constant.

Proof. We will construct the graph $F(n)$ as follows:

(i) The vertex set is the disjoint union of $A$ and $B$ where $|A| = 2n \log \log n / \log n$ and $|B| = 2n$.
(ii) Every vertex $v$ in $A$ is adjacent to all vertices in $V(F(n)) - \{v\}$.
(iii) The subgraph of $F(n)$ induced by $B$ is the graph, as described in Claim 3, which has $4n^2 \log \log n / \log n$ edges and contains all graphs with $2n$ vertices and degree at most $d$.

It is easy to see that $F(n)$ has at most $10n^2 \log \log n / n^2$ edges. Let $G$ be an arbitrary graph on $n$ edges. $G$ has at most $2n \log \log n / \log n$ vertices with degree more than $\log n / \log \log n$. These vertices will be embedded in $A$. The remaining part of the graph will then be embedded in $B$ as guaranteed by Claim 3.

This completes the proof of Claim 4.

Remark. If instead of using the result of Hajnal and Szemerédi, we use the simple fact that a graph on $n$ vertices and maximum degree $d$ can be $2(d+1)$ colored so that each color class has size at most $n/d$, then the resulting bound will differ from the one presented by a constant factor.

4. Universal graphs for planar graphs

We will use the following theorem to give an upper bound of $n^{3/2}$ for the universal graphs which contain all planar graphs on $n$ edges.

Separator Theorem (Lipton and Tarjan [6]). Let $G$ be any planar graph with $n$ vertices. The vertices of $G$ can be partitioned into three sets, $A$, $B$, $C$ such that no edge joins a vertex in $B$ with a vertex in $C$, neither $B$ and $C$ contain more than $n/2$ vertices, and $A$ contains no more than $2\sqrt{2n}/(1 - \sqrt{2/3})$ vertices.

Let $G(n)$ denote the graph constructed as shown in Fig. 1. The vertices of $G(n)$ can be partitioned into three parts, $X$, $Y$ and $Z$ where $|X|=$