Math 261
Midterm take home
Due: 1pm, Wednesday, May 12, 2010

You are only required to do three of the following four problems:

1. For two graphs $G = (V, E)$ and $H = (V', E')$, the box-product graph $G \square H$ denote the graph with vertex set $\{(u, v) : u \in V, v \in V'\}$ and edge set
   $\{(u, u'), (v, v') : (u = v \text{ and } \{u', v'\} \in E') \text{ or } (u' = v' \text{ and } \{u, v\} \in E)\}$. For the cycle $C_n$ on $n$ vertices, compute the number of spanning trees in $C_n \square C_n$. (Find a simple expression for your answer, if possible.)

2. In a connected graph $G$, we consider a subset of vertices, denoted by $S$. Suppose the induced subgraph on $S$ is connected. The Green function $G_S$ is defined to be the inverse of the Laplacian $L_S$. Recall that $L = I - D^{-1/2}AD^{-1/2}$. Now, we consider a cycle $C_{n+1}$ with vertices $0, 1, \ldots, n$ and vertex $j$ is adjacent to vertex $j+1$ for all $j$ modulo $n+1$. Choose $S = \{1, 2, \ldots, n\}$. Show that for $x \leq y$,
   $$G_S(x, y) = \frac{2}{n+1} x(n+1-y).$$

3. For a connected induced subgraph $S$ of a graph $G$ and for a real $t \geq 0$, the Dirichlet heat kernel of $S$ is defined by
   $$H_t = e^{-tL_S} = I - tL_S + \frac{t^2}{2!}L_S^2 + \ldots$$
   Show that
   $$G_S(x, y) = \int_0^\infty H_t(x, y) dt.$$

4. In a connected graph $G$, let $Z$ denote the lazy random walk $Z = (I + P)/2$ where $P$ is the transition probability matrix. For a seed $s$ (as a probability distribution) and a jumping constant $\alpha$, $0 < \alpha < 1$, the PageRank $p = pr_{\alpha, s}$ is defined to be the unique vector satisfies the following recurrence:
   $$p = \alpha s + (1-\alpha)pZ.$$
   For $\beta > 0$, we consider the $\alpha$-Green function to be the inverse of $L_\beta = \beta I + L$. Namely,
   $$L_\beta G_\beta = G_\beta L_\beta = I.$$
   Show that
   $$pr_{\alpha, s} = \beta sD^{-1/2}G_\beta D^{1/2}$$
   where $\beta = 2\alpha/(1-\alpha)$. 

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