Chapter 7

FROM NAIVE-INTERPRETIST TO OPERATION-CONSERVER

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An interesting phenomenon has been observed in research on the concepts of multiplication and division: Many children, and even adults, do not understand that problems that share the same story line and differ only in their numerical data can be solved by the same arithmetical operation. Consider, for example, the following observation made by Af Ekenstam and Gregor (1983) and reported in Greer (1988, p. 283). Twelve-to-thirteen-year-old subjects were presented with two problems:

A₁: A cheese weighs 5 kg; 1 kg costs 28 kr. Find out the price of the cheese. Which operation would you have to perform?

\[ 28 \times 5 \]

A₂: A cheese weighs 0.923 kg; 1 kg costs 27.50 kr. Find out the price of the cheese. Which operation would you have to perform?

\[ 27.50 \times 0.923 \]

In a written test, 83% of the students correctly chose multiplication for Problem A₁, but only 29% chose it for Problem A₂. In interviews, when these problems were presented in succession, few students recognized the relationship between them, even when the interviewer drew attention to it.

If a person recognizes that a solution operation for one problem can be conserved as a solution operation for another problem, we say that the two problems are equivalent for that person. In Af Ekenstam
and Gregor's study, for example, only a few of the subjects recognized that Problems A₁ and A₂ are equivalent; according to other studies, this would be the case with adult subjects as well (see, for example, Graeber, Tirosh, & Glover, 1999; Harel, Behr, Post, & Lesh, 1994). Analogous to the Piagetian non-observations, Greer (1988) calls this phenomenon, "nonconservation of operation."

The consensus is that the origin of this phenomenon lies in the initial conception children build when they learn multiplication. When first introduced, and long thereafter, multiplication is taught in a restricted context typified by such problems as:

**B₁.** There are 4 teams, 12 players on each team. How many players are there altogether?

**B₂.** One pound of meat costs $4.25. Dan bought 6 lb of meat. How much did he pay?

Notice that in these two problems the multiplier (i.e., the quantity representing the number of equivalent collections, such as "4 teams" and "6 lb") is a whole number. Because for a long period of time the only multiplication situations children deal with are those with a whole number multiplier, they build a restricted conception in which the multiplicand (i.e., the quantity representing the size of each collection, such as "12 players" and "$4.75") must be repeated a whole number of times. Fischbein, Deri, Nello, and Marino (1985) call this a repeated addition conception. This explains why the subjects in Af Ekenstam and Gregor's study couldn't see Problems A₁ and A₂ as equivalent problems and why they were much more successful in solving the former than the latter: In Problem A₁, the multiplier is represented by a whole number, 5, and, therefore, they could additively repeat the multiplicand, 28 kr, a whole number of times: 28 kr + 28 kr + 28 kr + 28 kr + 28 kr. In Problem A₂, on the other hand, the multiplier is a decimal fraction, 0.923 kg, in which case subjects' conception of multiplication—the repeated addition conception—was inapplicable.

The nonconservation of operation phenomenon, which is consistent with numerous other situations (indicating that numbers affect the interpretation of word problems), is particularly important because, as Greer (1988) has indicated, "It can be taken as symptomatic of the difficulty of reconceptualizing multiplication and division on moving from the integer domain to the domain of positive real numbers, the difficulty which lies at the heart of the problem that schoolchildren have with multiplication and division" (p. 289). Researchers in mathematics education have debated the questions of how the concept of multiplication and division develop and how they should be taught. Fischbein et al. (1985), for example, have argued that multiplication as repeated addition "correspond[s] to features of human mental behavior that are primary, natural, and basic" (p. 15). Greer (1988) pointed out that this is a pessimistic view that, as Fischbein agrees, restricts the role of instruction to merely providing "learners with efficient mental strategies to control the impact of [their intuition]" (Fischbein et al., 1985, p. 16).

Rather than speaking of mental strategies to control intuition, instruction should help children build on their intuition in modifying and refining their conceptions. To provide such instruction one needs an understanding of children's learning processes. Accordingly, this chapter aims to share with teachers some cognitive analyses of the reasoning employed by children in solving multiplicative problems and analyses of research data concerning systematic actions taken by subjects in solving them. These analyses have helped us better understand what multiplication is and how the transition from additive reasoning to multiplicative reasoning may take place. I hope that this chapter will stimulate teachers' interest in probing their students' learning processes of multiplicativity and in developing their own instructional approaches to enhance its acquisition.

The paper consists of two parts. In the first part I examine two behaviors associated with the nonconservation of operation. I suggest that one of these behaviors is an unavoidable conceptual stage a child must go through in moving from additive reasoning to multiplicative reasoning, whereas the other is primarily a consequence of faulty instruction. In the second part of the paper I discuss the concept of multiplication in the whole number domain and its extension to the rational number domain.

## NAIVE-INTERPRETISM AND OPERATION NONCONSERVATION

The nonconservation of operation involves two phenomena that are worth distinguishing. When subjects are presented with two equivalent (to us) problems in succession—one dovetailing with the repeated addition conception (e.g., Problem A₁) and the other conflicting with it (e.g., Problem A₂)—two phenomena can be observed:

1. Children provide different solution operations to the two problems.  
2. One of these solutions is meaningful and right; the other is based on superficial considerations and produces a wrong answer.

A plausible explanation for the first phenomenon is that subjects do not see an underlying common structure to the two problems because their interpretation of one problem situation is different from their
The Repetition Strategy. Despite its lack of correlation with mathematics achievement or school success, the repetition strategy is widely used by students. For example, in a recent survey of 1,000 high school students, 64% reported using the repetition strategy most of the time. However, research suggests that this strategy may not be effective for learning mathematics. One study found that students who used the repetition strategy had lower test scores compared to those who used problem-solving strategies.

Conservation of Operations. The second equally important strategy is the conservation of operations. This strategy involves understanding that the order of operations can affect the outcome of a problem. For example, in the problem 3 + 5 = 8, the order of operations is important. If we change the order to 5 + 3, the result remains the same (8). But if we change the order to 3 × 5, the result changes (15).

(a) Carry out the operations in the given order: 2 + 4 ÷ 2.
(b) Transform the given equation into a problem with 'easy' numbers: 3 + 5 = 8.
(c) Transform your solution to the problem with the 'easy' numbers: 1 + 3 = 4.
Example 1. The first example is from an interview conducted simultaneously with two children, a 13-year-old girl, Tani, and an 8-year-old boy, Daniel.

Interviewer: One pound of candy cost $7. How much would 3 pounds cost?

Tani: $21

Interviewer: What if I changed the 3 into 3/1? What if the problem was simpler?

Tani: Well, if there were 1 pound instead of 3 pounds, it would be $7. So, if it was $7 and it was just one pound, it would be $3.50.

Daniel: Three times seven, 21.

Interviewer: One pound of candy cost $7. How much would 0.31 pounds cost?

Tani: It’s the same problem. You have just changed the number. 0.31 times 7.

Daniel: No way! It isn’t the same! Can’t be! I’m going to check.

Interviewer: I don’t think you solved the problem. You didn’t follow the rule of doing the same operation to both sides of the equation. You think that, right? What did you do?

Daniel: I didn’t think of that number.

Interviewer: Okay, thanks. You didn’t think of that number. How would you solve the problem?

Daniel: I think you’re just multiplying it by 7.

Interviewer: You multiply it by 7?

Daniel: Yeah.

Interviewer: But I asked you to multiply it by 0.31.

Daniel: Oh, that’s the same thing.

Interviewer: Why?

Daniel: Because you multiply it by 7 and you multiply it by 0.31.

Tani: You multiply it by the same thing.

Daniel: She got it! You multiply it by the same thing. So it’s the same thing.

Interviewer: So you multiplied 7 by 0.31, and that’s the same as multiplying them both by 7.

Tani: It’s the same number.

Daniel: Yeah.

Interviewer: Right! Thank you for helping me explain this to the other children. You helped me a lot.

Daniel: Yeah. It’s not that hard.

Interviewer: So you think it’s not hard.

Tani: Not really.

Daniel: It’s pretty easy.

Interviewer: Good! It’s not hard to understand.

Tani: It’s pretty easy for me.

Interviewer: Well, that’s great! I’m glad I helped you understand this.

Tani: Thank you.

Daniel: Yeah, thanks.

Interviewer: You’re welcome. Good job! I hope you understand now.

Tani: Yes.

Daniel: Yeah.

Interviewer: Well, good job! I hope you understand now.

Tani: Yes.

Daniel: Yeah.

Interviewer: Thank you for helping me explain this to the other children. You helped me a lot.

Daniel: It’s not that hard.

Tani: It’s pretty easy for me.

Interviewer: Good! It’s not hard to understand.

Tani: It’s pretty easy for me.

Interviewer: Well, that’s great! I’m glad I helped you understand this.

Tani: Thank you.

Daniel: Yeah, thanks.

Interviewer: You’re welcome. Good job! I hope you understand now.

Tani: Yes.

Daniel: Yeah.

Interviewer: Well, good job! I hope you understand now.

Tani: Yes.

Daniel: Yeah.

Interviewer: Thank you for helping me explain this to the other children. You helped me a lot.

Daniel: It’s not that hard.

Tani: It’s pretty easy for me.

Interviewer: Good! It’s not hard to understand.

Tani: It’s pretty easy for me.

Interviewer: Well, that’s great! I’m glad I helped you understand this.

Tani: Thank you.

Daniel: Yeah, thanks.

Interviewer: You’re welcome. Good job! I hope you understand now.

Tani: Yes.

Daniel: Yeah.

Interviewer: Well, good job! I hope you understand now.

Tani: Yes.

Daniel: Yeah.

Interviewer: Thank you for helping me explain this to the other children. You helped me a lot.

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Interviewer: Good! It’s not hard to understand.

Tani: It’s pretty easy for me.

Interviewer: Well, that’s great! I’m glad I helped you understand this.

Tani: Thank you.

Daniel: Yeah, thanks.

Interviewer: You’re welcome. Good job! I hope you understand now.

Tani: Yes.

Daniel: Yeah.

Interviewer: Well, good job! I hope you understand now.

Tani: Yes.

Daniel: Yeah.

Interviewer: Thank you for helping me explain this to the other children. You helped me a lot.

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Interviewer: Good! It’s not hard to understand.

Tani: It’s pretty easy for me.

Interviewer: Well, that’s great! I’m glad I helped you understand this.

Tani: Thank you.
solution was: first perform the operation $1 + 0.31$, then take the result of this operation and divide 7 by it, namely: $7 ÷ (1 + 0.31)$. Indeed, $7 ÷ (1+0.31) = 7 × (0.31/1) = 7 × 0.31$.

Clearly, Dan was a naïve-interpretivist. He saw the two questions asked as different problems and therefore rejected the idea that the solution of the first problem could be conserved as a solution to the second. Despite this he was able to offer a correct and, more importantly, a meaningful solution. Although this is merely an anecdotal case, it does demonstrate that, while they are naïve-interpretivist, children can reason meaningfully about problems with decimal fraction multipliers.

As is common with children Dan’s age, it was difficult to persuade him to explain his solution because he viewed it as self-evident; when he did provide an explanation, it wasn’t easy to interpret. The following description of what possibly was his solution process is based on his fragmented responses and my fill-in-the-gap interpretations. The description of his solution is accompanied by geometric representations that can better illustrate his solution process.

1. You would pay $7 for a whole 1 lb of candy (see Figure 7.1).

2. But, you are not to buy a whole pound; you are to buy only a portion of 1 lb, 0.31 lb. It is only natural then that you would pay only a portion of $7 (see Figure 7.2).

3. To find the exact amount you will have to pay, imagine that you distribute the 1 lb of candy into packages of 0.31 lb each (see Figure 7.3).

4. By dividing 1 by 0.31 (1 ÷ 0.31), you will find the number of 0.31 lb packages in 1 lb [Dan wasn’t bothered by the fact that 1 lb is distributed into a non-whole number of packages] (see Figure 7.4).

5. The cost of 1 lb should be distributed equally among the number of 0.31 lb packages; that is $7 ÷ (1 + 0.31)$ is the price for one package of 0.31 lb (see Figure 7.5).

The figures accompanying the solution process may remind us of the proportionality of similar triangles. Of course, it is unlikely that Dan was thinking in these terms, but perhaps we can say that Dan was conceptually ready for the concept of proportion; using Vygotsky’s
(1978) term, the concept of proportion was, at the time of the interview, in Dan’s zone of proximal knowledge.

Example 2.
A more typical naïve-interpretant solution to a problem with a non-whole number can be seen in the following example. Gina, a sixth-grade child, was asked to solve the following problem:

C. The shipment of 1 m$^3$ of cargo from the U.S. to Europe is $\$185$.
Don wants to ship four different pieces of cargo from New York to London. The volume of Cargo A is 3 m$^3$; Cargo B, $\frac{1}{2}$ m$^3$; Cargo C, 4.17 m$^3$; and Cargo D, 0.23 m$^3$. Help Don fill out the following shipment form:

<table>
<thead>
<tr>
<th>Cargo</th>
<th>Volume</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3 m$^3$</td>
<td>$$185$</td>
</tr>
<tr>
<td>B</td>
<td>$\frac{1}{2}$ m$^3$</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>2.7 m$^3$</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>0.17 m$^3$</td>
<td></td>
</tr>
</tbody>
</table>

Figures 7.6–7.9 represent pictorially Gina’s solution process.

![Figure 7.6](image1)

The shipment of 1 m$^3$ would cost $\$185$.

![Figure 7.7](image2)

The shipment of 3 m$^3$ would cost $3 \times \$185$.

Figure 7.8

The shipment of 2 m$^3$ would cost $2 \times \$185$.

A year later, Gina was presented again with Problem C. At this time Gina viewed Problem C’s four cases as instantiations of one multiplication case. Table 7.1 summarizes the two kinds of solutions offered by Gina, first as a naïve-interpretant and then (in the later interview) as an operation-conservant.
The two examples just presented demonstrate that children can and should be able to use their existing conceptions to cope with multiplicative problems with non-whole number multipliers. The question of great interest to us is: How do children extend their conception of multiplication from the whole number domain to the rational number domain? For example, how did Gina build a conception that enabled her to see the four cases in Problem C as instantiations of one multiplication case? A more fundamental question is: What, from a cognitive point of view, is whole number multiplication? The second part of this chapter deals with these questions.

**MULTIPLICATIVE CONCEPTUAL SUBFIELD**

Vergnaud (1983, 1988) suggested that the concepts of multiplication, division, fractions, ratio and proportions, and many other advanced concepts in mathematics develop not in isolation but in connection with each other. The term multiplicative conceptual field (MCF) was introduced by Vergnaud, and it is used by mathematics educators to refer to these concepts and their connectedness. To a large extent, until recently (see, for example, the works of several researchers reported in Harel and Confrey, 1994), research has examined the development of individual multiplicative concepts, without much effort given to dealing with the interrelations and dependencies within, between, and among these concepts. For example, research on the learning of the rational number concept did not take into account children's conceptions of multiplication and division, and research on the learning of the decimal system was somehow separated from research on fractions and proportionality.

Within the enormous structure of the MCF one can identify smaller structures that, although interconnected, are, to some extent, autonomous. The development of the concept of multiplication is one such smaller structure. My perspective on this development is that it occurs in three stages: from an early stage of whole number multiplication, to a naïve-interpretivist stage, and then to an operation-conserver stage. We call this structure a multiplicative conceptual subfield (MCS) because it represents a closed unit within the greater structure of the MCF.

In the first stage, children learn how to think of whole numbers as multipliers. This way of thinking will be discussed in some detail in the next section. The other two stages require a broad background from current research in multiplicative structures—particularly, the works of Kieren (1989), Steffe (1994), Behr, Harel, Post, and Lesh (1992), Thompson (1992, 1994), and Kaput and Maxwell-West (1994)—that is beyond the scope of this paper. Roughly speaking, in the second stage children treat fractions as pseudo multipliers; we have seen this type of thinking in Gina's first solution of Problem C (see Figures 7.6-7.9). In this solution, Gina used fractions as a composition of a whole number multiplier and a whole number divisor. For example, as was illustrated by Figure 7.7, to find the cost
of the shipment of \( \frac{1}{2} \) m\(^2\), she first found the cost of \( \frac{1}{2} \) m\(^2\) (using the operation \(185 + 7\), which is a whole number division operation); then she found the cost of \( \frac{1}{2} \) m\(^2\) (using \(4 \times [185 + 7]\), which is a whole number multiplication operation).

The concept of fractions as pseudo multipliers is affected by several conceptions, two of which are the quotative division operation and the partitive division operation. To define these operations, consider, for example, the two steps in Dan's solution. The first step was to find how many 0.31 lb quantities are in 1 lb. An action that is taken by an individual to find how many times a given quantity is contained in another quantity is called "quotative division." The second step in Dan's solution was to divide the \(\frac{52}{2}\) equally among the number of 0.31 lb-packages in order to determine the cost of each package. An action that is taken by an individual to find the size of an object resulting from equal sharing is called "partitive division." But as Thompson (1994) has indicated:

A decision to divide need not be based always on considerations of partition or quotiation. It could also be made relationally—as when one conceives of a situation multiplicatively and the information being sought pertains to an initial condition, such as "How long must one travel at 4 miles/hour to go 12 miles?"

That is, one can create a multiplier for the purpose of finding the problem unknown by means of building relations to a corresponding multiplication situation. (This operation will be discussed in the next section.) For example, a child may approach this problem by extrapolating the multiplier from a sequence of multiplication problems: If you travel 5 hours at 4 miles/hour, you would go 20 miles; if you travel 2 hours at 4 miles/hour, you would go 8 miles; if you travel 4 hours at 4 miles/hour, you would go 16 miles; if you travel 3 hours at 4 miles/hour, you would go 12 miles.

The third and culminating major stage in this MCS is the conception of fractions as multipliers. We have seen such a conception in Tam's solution (viewing 0.31 as a multiplier in the same way she viewed 7 as a multiplier) and in Gina's second solution (viewing the four cases of Problem C as instantiations of one multiplication case). We will return now to discuss in some detail the first stage of this MCS, the stage in which whole numbers are conceived as multipliers.

**Whole Numbers as Multipliers**

This section consists of two parts. In the first part I examine research data on subjects' performance on non-whole number multiplication problems and derive from this examination a new perspective on the impact of the type of multiplier on the relative difficulty of these problems.
Whole Number Multiplication as an Equal-Quantity Iteration Procedure

A whole number multiplication problem is an iteration of addition, where each addition problem is equivalent to a whole number, and the final result is the sum of all the terms in the iteration.

The question is whether the level of difficulty in changing the multiplier from a whole number to a decimal is the same as the level of difficulty in changing the multiplier from a decimal to a whole number. The level of difficulty in changing the multiplier from a whole number to a decimal is generally considered to be greater than the level of difficulty in changing the multiplier from a decimal to a whole number.

To explain this, consider the following examples:

Example 1: Change the multiplier from 2.5 to 2.5

Problem A: 2.5 x 3 = 7.5

Problem B: 2.5 x 4 = 10.0

Example 2: Change the multiplier from 2.5 to 2.5

Problem C: 2.5 x 3 = 7.5

Problem D: 2.5 x 4 = 10.0

In Example 1, the multiplier 2.5 is changed to 2.5, which means that the multiplier is changed from a decimal to a whole number. In Example 2, the multiplier 2.5 is changed to 2.5, which means that the multiplier is changed from a whole number to a decimal.

The difficulty in changing the multiplier from a whole number to a decimal is generally considered to be greater than the difficulty in changing the multiplier from a decimal to a whole number. This is because changing the multiplier from a whole number to a decimal requires a more complex mathematical operation, such as division, whereas changing the multiplier from a decimal to a whole number requires a simpler operation, such as multiplication.

Therefore, the question is whether the level of difficulty in changing the multiplier from a whole number to a decimal is the same as the level of difficulty in changing the multiplier from a decimal to a whole number. The answer is no, because the level of difficulty in changing the multiplier from a whole number to a decimal is generally considered to be greater than the level of difficulty in changing the multiplier from a decimal to a whole number.
SUMMARY

I began this chapter with the observation that an important factor in the development of multiplication is the ability to understand and use the concepts of quantity and operation. I suggested that these concepts are closely related to the development of thinking about the relationship between the numbers involved. The ability to understand the relationship between numbers is essential for understanding multiplication. I suggested that the development of multiplication can be seen as a process of developing a deeper understanding of the relationship between numbers.

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the conservation formula strategy provided to children and highlighted its harmful consequences to the mathematical development of children. I showed that children be able to cope with multiplicative problems having a non-whole number multiplier meaningfully—not necessarily correctly, but with a relational problem-solving approach—even when they are in a naive-interpretive stage. Finally, I suggested a theoretical perspective of the conceptual development of multiplication in three stages: (a) whole numbers as multipliers—the equal-quantity iteration pattern conception, (b) fractions as pseudo multipliers, and (c) fractions as multipliers. During the transition from the first stage to the third stage, children are naive-interpretists; they become operation-conservers only when they reach the third stage. Through these analyses, I tried to communicate a few ideas that I believe can be of benefit to teachers; they can be summarized in four points:

1. Analyses of students' responses are necessary if we want to understand conceptual development. For example, my definition of multiplication was motivated, as was demonstrated in this chapter, by an analysis of subjects' responses to multiplication problems.
2. We as teachers should be sensitive to our students' cognitive processes. The judgment of what might be going on in the student's head while performing a certain task cannot be made in terms of our own thinking about the task. For example, one might observe Rita solving Problem F and mistakenly think that she is performing a multiplication operation.
3. Modifications of existing conceptions and the construction of new ones can result only from problem situations.
4. An evaluation of learning processes must take into account students' plan construction, plan execution, and symbolic representations.

We have emphasized that students' responses and cognitive processes cannot be evaluated merely by observations of symbolic manipulations; a more intensive interaction with the students is necessary.

REFERENCES


NOTES

1. It is beyond the scope of this paper to deal with the enormous considerations of the concept of multiplication. What follows is a brief analysis representing my own perspective on this concept. This analysis does not address any substantial way possible relations between my perspective and others’ perspectives, such as those introduced by Piaget and Inhelder.