HW2 Solutions

MATH 20D Fall 2013
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Checklist:

Section 2.6: 1, 3, 6, 8, 10, 15, [20, 22]
Section 3.1: 1, 2, 3, 9, 16, 18, 20, 23
Section 3.2: 1, 2, 3, 7, 9, 13, 14, 24
Section 3.3: 8, 9, 15, 19, 21, 22, 27, 29, [34, 36]
Determine whether each of the equations in problems 2.6.1 thru 2.6.10 is exact. If it is exact, find the solution.

**Problem 2.6.1.** \((2x + 3) + (2y - 2)y' = 0\)

*Proof.* Notation as in Theorem 2.6.1 from the book. Then \(M(x, y) = 2x + 3\) and \(N(x, y) = 2y - 2\). Since \(M_y = N_x = 0\), this equation is exact by the theorem. So we should be able to yield an implicit solution \(\psi(x, y) = c\).

Now integrate \(M\) with respect to \(x\) while holding \(y\) constant:

\[
\int (2x + 3) dx = x^2 + 3x \implies \psi(x, y) = x^2 + 3x + h(y)
\]

Then integrate \(N\) with respect to \(y\) while holding \(x\) constant:

\[
\int (2y - 2) dy = y^2 - 2y \implies \psi(x, y) = y^2 - 2y + g(x)
\]

Comparing the previous two lines, we get \(\psi(x, y) = x^2 + 3x + y^2 - 2y\). Hence the (implicit) solution is \(x^2 + 3x + y^2 - 2y = c\). ■

**Problem 2.6.3.** \((3x^2 - 2xy + 2) + (6y^2 - x^2 + 3)y' = 0\)

*Proof.* \(M(x, y) = 3x^2 - 2xy + 2\) and \(N(x, y) = 6y^2 - x^2 + 3\). Since \(M_y = -2x = N_x\), the equation is exact.

Now integrate \(M\) with respect to \(x\) while holding \(y\) constant:

\[
\int (3x^2 - 2xy + 2) dx = x^3 - x^2y + 2x \implies \psi(x, y) = x^3 - x^2y + 2x + h(y)
\]

Then integrate \(N\) with respect to \(y\) while holding \(x\) constant:

\[
\int (6y^2 - x^2 + 3) dy = 2y^3 - x^2y + 3y \implies \psi(x, y) = 2y^3 - x^2y + 3y + g(x)
\]

Comparing the previous two lines, we get \(\psi(x, y) = x^3 - x^2y + 2x + 2y^3 + 3y\). Hence the (implicit) solution is \(x^3 - x^2y + 2x + 2y^3 + 3y = c\). ■

**Problem 2.6.6.**

\[
\frac{dy}{dx} = -\frac{ax - by}{bx - cy}
\]
Proof. We rewrite the equation as \((ax - by) + (bx - cy) \frac{dy}{dx} = 0\), so \(M(x, y) = ax - by\) and \(N(x, y) = bx - cy\). Since \(M_y = -b\), \(N_x = b\), the equation is not exact when \(b \neq 0\).

When \(b = 0\), the equation becomes \(ax - cy \frac{dy}{dx} = 0\), which is not only exact, but also separable:

\[
ax - cy \frac{dy}{dx} = 0
\]

\[
\frac{ax}{cy} \frac{dy}{dx} = \frac{cy}{ax}
\]

\[
\Rightarrow \int ax \, dx = \int cy \, dy
\]

\[
\frac{a}{2}x^2 = \frac{c}{2}y^2
\]

In this case the solution is \(\frac{a}{2}x^2 - \frac{c}{2}y^2 = \text{constant}\).

Problem 2.6.8. \((e^x \sin y + 3y) - (3x - e^x \sin y)\frac{dy}{dx} = 0\)

Proof. \(M(x, y) = e^x \sin y + 3y\) and \(N(x, y) = e^x \sin y - 3x\). Since \(M_y = e^x \cos y + 3\) and \(N_x = e^x \cos y - 3\) are not equal, the equation is not exact.

Note: Be careful of the negative sign in front of \(- (3x - e^x \sin y)!\)

Problem 2.6.10. \((y/x + 6x) + (\ln x - 2)\frac{dy}{dx} = 0\)

Proof. \(M(x, y) = y/x + 6x\) and \(N(x, y) = \ln x - 2\). Since \(M_y = \frac{1}{x} = N_x\), the equation is exact.

Now integrate \(M\) with respect to \(x\) while holding \(y\) constant:

\[
\int (y/x + 6x) \, dx = y \ln x + 3x^2 \Rightarrow \psi(x, y) = y \ln x + 3x^2 + h(y)
\]

Then integrate \(N\) with respect to \(y\) while holding \(x\) constant:

\[
\int (\ln x - 2) \, dy = y \ln x - 2y \Rightarrow \psi(x, y) = y \ln x - 2y + g(x)
\]

Comparing the previous two lines, we get \(\psi(x, y) = y \ln x - 2y + 3x^2\). Hence the (implicit) solution is \(y \ln x - 2y + 3x^2 = c\).

In Problem 2.6.15 find the value of \(b\) for which the given equation is exact, and then solve it using that value of \(b\).

Problem 2.6.15. \((xy^2 + bx^2y) + (x + y)x^2y' = 0\)
Proof. \(M(x, y) = xy^2 + bx^2y\) and \(N(x, y) = (x+y)x^2\). Since \(M_y = 2xy + bx^2\), \(N_x = 3x^2 + 2xy\), the equation is exact when \(b = 3\), not exact otherwise.

Given \(b = 3\), integrate \(M\) with respect to \(x\) while holding \(y\) constant:

\[
\int (xy^2 + 3x^2y)dx = \frac{1}{2}x^2y^2 + x^3y \implies \psi(x, y) = \frac{1}{2}x^2y^2 + x^3y + h(y)
\]

Then integrate \(N\) with respect to \(y\) while holding \(x\) constant:

\[
\int ((x + y)x^2)dy = x^3y + \frac{1}{2}x^2y^2 \implies \psi(x, y) = x^3y + \frac{1}{2}x^2y^2 + g(x)
\]

Comparing the previous two lines, we get \(\psi(x, y) = x^3y + \frac{1}{2}x^2y^2\). Hence the (implicit) solution is \(2x^3y + x^2y^2 = c\). ■

In each of Problems 2.6.20 thru 2.6.22 show that the given equation is not exact but becomes exact when multiplied by the given integrating factor. Then solve the equation.

**Optional Problem 2.6.20.**

\[
\left(\frac{\sin y}{y} - 2e^{-x} \sin x\right) + \left(\frac{\cos y + 2e^{-x} \cos x}{y}\right)y' = 0, \quad \mu(x, y) = ye^x
\]

Proof. \(M_y = y^{-1} \cos y - y^{-2} \sin y\) and \(N_x = -2e^{-x}(\cos x + \sin x)/y\).

Multiply both sides by the integrating factor \(\mu(x, y) = ye^x\), the given equation could be written as:

\[
(e^x \sin y - 2y \sin x)dx + (e^x \cos y + 2 \cos x)dy = 0 \quad (I)
\]

Let \(\tilde{M} = \mu M\) and \(\tilde{N} = \mu N\). Observe that \(\tilde{M}_y = \tilde{N}_x\), hence the above ODE (I) is exact.

Integrate \(\tilde{N}\) with respect to \(y\) while holding \(x\) constant:

\[
\int (e^x \cos y + 2 \cos x)dy = e^x \sin y + 2y \cos x \implies \psi(x, y) = e^x \sin y + 2y \cos x + h(x)
\]

We can differentiate this with respect to \(x\), \(\psi_x = e^x \sin y - 2y \sin x + h'(x)\). Setting \(\psi_x = \tilde{M}\), we find that \(h'(x) = 0\), hence \(h(x) = 0\) is feasible. Hence the solution of the given equation is defined implicitly by \(e^x \sin y + 2y \cos x = c\). ■

**Optional Problem 2.6.22.** \((x + 2) \sin y + (x \cos y)y' = 0\) \quad \mu(x, y) = xe^x

Proof. \(M_y = (x + 2) \cos y\) and \(N_x = \cos y\).

Multiply both sides by the integrating factor \(\mu(x, y) = xe^x\), the given equation could be written as:

\[
x e^x(x + 2) \sin ydx + x^2 e^x \cos ydy = 0 \quad (II)
\]

Let \(\tilde{M} = \mu M\) and \(\tilde{N} = \mu N\). Observe that \(\tilde{M}_y = \tilde{N}_x = xe^x(x + 2) \cos y\), hence the above ODE (II) is exact.
Integrate $\tilde{N}$ with respect to $y$ while holding $x$ constant:

$$\int (x^2 e^x \cos y) \, dy = x^2 e^x \sin y \implies \psi(x, y) = x^2 e^x \sin y + h(x)$$

We can differentiate this with respect to $x$, $\psi_x = e^x (x^2 + 2x) \sin y + h'(x)$. Setting $\psi_x = \tilde{M}$, we find that $h'(x) = 0$, hence $h(x) = 0$ is feasible. Hence the solution of the given equation is defined implicitly by $x^2 e^x \sin y = c$. $\blacksquare$

Section 3.1

Reminder: The methods of this section only applies to the case where the roots of the characteristic equation are real and different. The case where the roots are complex are treated in Section 3.3, while Section 3.4. discusses the case where both roots are equal.

In each of Problems 3.1.1 thru 3.1.8 find the general solution of the given differential equation.

**Problem 3.1.1.** $y'' + 2y - 3y = 0$

*Proof.* Let $y = e^{rt}$, so that $y' = re^{rt}$ and $y'' = re^{rt}$. Direct substitution into the differential equation yields $(r^2 + 2r - 3)e^{rt} = 0$. Canceling the exponential, the characteristic equation is $(r^2 + 2r - 3) = 0$. The roots of the equation are $r = -3, 1$. Hence the general solution is $y = c_1 e^t + c_2 e^{-3t}$. $\blacksquare$

**Problem 3.1.2.** $y'' + 3y' + 2y = 0$

*Proof.* Let $y = e^{rt}$. It follows that $r$ must satisfy the characteristic equation $(r^2 + 3r + 2) = 0$. The roots of the equation are $r = -2, -1$. Hence the general solution is $y = c_1 e^{-2t} + c_2 e^{-t}$. $\blacksquare$

**Problem 3.1.3.** $6y'' - y' - y = 0$

*Proof.* Let $y = e^{rt}$. It follows that $r$ must satisfy the characteristic equation $6r^2 - r - 1 = 0$. The roots of the equation are $r = -\frac{1}{3}, \frac{1}{2}$. Hence the general solution is $y = c_1 e^{-\frac{1}{3}t} + c_2 e^{\frac{1}{2}t}$. $\blacksquare$

In each of Problems 3.1.9 through 3.1.16 find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior as $t$ increases.

**Problem 3.1.9.** $y'' + y' - 2y = 0$, $y(0) = 1$, $y'(0) = 1$
Proof. Let \( y = e^{rt} \). It follows that \( r \) must satisfy the characteristic equation \( (r^2 + r - 2) = 0 \). The roots of the equation are \( r = -2, 1 \). Hence the general solution is \( y = c_1 e^{-2t} + c_2 e^t \). Its derivative is \( y' = -2c_1 e^{-2t} + c_2 e^t \). Based on the first condition, \( y(0) = 1 \), we require that \( c_1 + c_2 = 1 \); the second condition \( y'(0) = 1 \) requires that \( -2c_1 + c_2 = 1 \). Solving the following system of linear equations:

\[
\begin{align*}
    c_1 + c_2 &= 1 \\
    -2c_1 + c_2 &= 1
\end{align*}
\]

we yield that \( c_1 = 0 \) and \( c_2 = 1 \). Hence the specific solution is \( y(t) = e^t \). It is clear that \( y \to \infty \) as \( t \to \infty \).

In problems 3.1.9 and 3.1.16 find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior as \( t \) increases.

**Problem 3.1.16.** \( 4y'' - y = 0, \quad y(-2) = 1, \quad y'(-2) = 1 \)

**Proof.** Let \( y = e^{rt} \). It follows that \( r \) must satisfy the characteristic equation \( (4r^2 - 1) = 0 \). The roots of the equation are \( r = \pm \frac{1}{2} \). Hence the general solution is \( y = c_1 e^{\frac{1}{2}t} + c_2 e^{-\frac{1}{2}t} \). Since the initial condition is specified at \( t = -2 \), it is more convenient to write \( y = d_1 e^{\frac{1}{2}(t+2)} + d_2 e^{-\frac{1}{2}(t+2)} \), where \( d_1 \) and \( d_2 \) are constants. The derivative is \( y' = \frac{1}{2}d_1 e^{\frac{1}{2}(t+2)} - \frac{1}{2}d_2 e^{-\frac{1}{2}(t+2)} \). The initial conditions, \( y(-2) = 1, \ y'(-2) = 1 \) require that \( d_1 + d_2 = 1 \) and \( d_1/2 - d_2/2 = 1 \). Solving for the coefficients one can see that \( d_1 = -\frac{1}{2} \) and \( d_2 = \frac{3}{2} \). Hence the specific solution is \( y(t) = -\frac{1}{2}e^{\frac{1}{2}(t+2)} + \frac{3}{2}e^{-\frac{1}{2}(t+2)} = -\frac{1}{2e}e^{\frac{1}{2}t} + \frac{3}{2e}e^{-\frac{1}{2}t} \).

Since the first summand in \( y \) goes to negative infinity and the second summand goes to zero as \( t \to \infty \), it follows that \( y \to -\infty \) as \( t \to \infty \).
Problem 3.1.18. Find a differential equation whose general solution is
\[ y = c_1 e^{2t} + c_2 e^{-3t}. \]

Proof. The form of this general solution suggests that such an equation to be second order homogeneous with constant coefficients. The powers \(2t\) and \(-3t\) imply that the equation should have characteristic equation \((r - 2)(r + 3) = r^2 + r - 6 = 0\). It is clear from here that the equation is \(y'' + y' - 6y = 0\). \(\blacksquare\)

Problem 3.1.20. Find the solution of the initial value problem
\[ 2y'' - 3y' + y = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{2}. \]

Then determine the maximum value of the solution and also find the point where the solution is zero.

Proof. Let \(y = e^{rt}\). It follows that \(r\) must satisfy the characteristic equation \((2r^2 - 3r + 1) = 0\).

The roots of the equation are \(r = 1, \frac{1}{2}\). Hence the general solution is \(y = c_1 e^t + c_2 e^{\frac{1}{2}t}\). The derivative is \(y' = c_1 e^t + \frac{1}{2} c_2 e^{\frac{1}{2}t}\). In order to satisfy the initial conditions we require that \(c_1 + c_2 = 2\) and \(c_1 + c_2/2 = \frac{1}{2}\). Solving for the coefficients, \(c_1 = -1\) and \(d_2 = 3\). Hence the specific solution is \(y = -e^t + 3e^{\frac{1}{2}t}\).

Recall that a maximum value, if not infinity, can be found at a stationary point, i.e. a point where \(y' = 0\). In the setting of this problem, solving \(y' = -e^t + \frac{3}{2} e^{\frac{1}{2}t} = e^{\frac{1}{2}t}(-e^{\frac{1}{2}t} + \frac{3}{2}) = 0\) gives a unique solution \(t_1 = \ln(9/4)\). Note that since \(e^{\frac{1}{2}t}\) is positive, the sign of \(y'\) is...
determined by the sign of \( z = -e^{\frac{1}{2}t} + \frac{3}{2} \). As \(-e^{\frac{1}{2}t}\) is a decreasing function, the sign of \( z \), thus \( y' \), is initially positive but becomes negative for \( t > \ln(9/4) \). This justifies that \( y \) takes its maximum value at \( \ln(9/4) \), which is \( 9/4 \).

To find the \( x \)-intercept, solve the equation \(-e^t + 3e^{\frac{1}{2}t} = 0\). The solution is readily found to be \( t_2 = \ln 9 \approx 2.1972 \).

In Problem 3.1.23 determine the values of \( \alpha \), if any, for which all solutions tend to zero as \( t \to \infty \); also determine the values of \( \alpha \), if any, for which all (nonzero) solutions become unbounded as \( t \to \infty \).

**Problem 3.1.23.** \( y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0 \)

**Proof.** The characteristic equation is \( r^2 - (2\alpha - 1)r + \alpha(\alpha - 1) = 0 \). Examining the coefficients, the roots are \( r = \alpha, \alpha - 1 \). Hence the general solution of the equation is \( y(t) = c_1e^{\alpha t} + c_2e^{(\alpha - 1)t} = e^{(\alpha - 1)t}(c_1e^t + c_2) \).

For the second part, in order to let \( y(t) = c_1e^{\alpha t} + c_2e^{(\alpha - 1)t} \) tend to zero as \( t \to \infty \), we require that the power in both exponentials be negative, which means that \( \alpha < 0 \).

For the third part, note that \( c_1e^t + c_2 \) always tend to \( \pm \infty \) when \( t \to \infty \), as long as \( e^{(\alpha - 1)t} \) does not tend to zero, we’re good since the product of the two are being considered. This is equivalent to saying that \( \alpha - 1 > 0 \), i.e. \( \alpha > 1 \).

**Section 3.2**

In each of Problems 3.2.1 through 3.2.6 find the Wronskian of the given pair of functions.

**Remark.** See Page 149 of the textbook for the definition of the Wronskian \( W(y_1, y_2) \).

**Problem 3.2.1.** \( e^{2t}, e^{-3t/2} \)

**Proof.**

\[
W(e^{2t}, e^{-3t/2}) = \begin{vmatrix} e^{2t} & e^{-3t/2} \\ 2e^{2t} & -\frac{3}{2}e^{-3t/2} \end{vmatrix} = -\frac{7}{2}e^{-\frac{t}{2}}
\]

**Problem 3.2.2.** \( \cos t, \sin t \)

**Proof.**

\[
W(\cos t, \sin t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1
\]

**Problem 3.2.3.** \( e^{-2t}, te^{-2t} \)
Proof.

\[ W(e^{-2t}, te^{-2t}) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix} = e^{-4t} \]

In each of Problems 3.2.7 through 3.2.12 determine the longest interval in which the given initial value problem is certain to have a unique twice differentiable solution. Do not attempt to find the solution.

Remark: For the following two problems, you might use the following theorem from page 146:

**Theorem 3.2.2.** Consider the initial value problem

\[ y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y'_0 \]  

where \( p, q, g \) are continuous on an open interval \( I \) that contains the point \( t_0 \). Then there is exactly one solution \( y = \phi(t) \) of this problem, and the solution exists throughout the interval \( I \).

**Problem 3.2.7.** \( ty'' + 3y = t, \quad y(1) = 1, \quad y'(1) = 2 \)

Proof. Write the equation as \( y'' + (3/t)y' = 1 \), so we can apply the theorem above. \( p(t) = 3/t \) is continuous for all \( t > 0 \). Since \( t_0 > 0 \), the initial value problem has a unique solution for all \( t > 0 \).

**Problem 3.2.9.** \( t(t-4)y'' + 3ty' + 4y = 2, \quad y(3) = 0, \quad y'(3) = -1 \)

Proof. Write the equation as \( y'' + (3/(t-4))y' + (4/(t-4))y = 2/t(t-4) \). The coefficients are not continuous at \( t = 0 \) and \( t = 4 \). Since \( t_0 \in (0, 4) \), the largest interval is \( 0 < t < 4 \).

**Problem 3.2.13.** Verify that \( y_1(t) = t^2 \) and \( y_2(t) = t^{-1} \) are two solutions of the differential equation \( t^2y'' - 2y = 0 \) for \( t > 0 \). Then show that \( y = c_1t^2 + c_2t^{-1} \) is also a solution of this equation for any \( c_1 \) and \( c_2 \).

Proof. \( y_1'' = 2 \). We see that \( t^2(2) - 2(t^2) = 0 \). \( y_2'' = 2t^{-3} \), with \( t^2(y_2'') - 2(y_2) = 0 \). Let \( y_3 = c_1t^2 + c_2t^{-1} \), then \( y_3'' + 2c_1 + 2c_2t^{-3} \). It is evident that \( y_3 \) is also a solution (Also See Theorem 3.2.2 from the book).

**Problem 3.2.14.** 14. Verify that \( y_1(t) = 1 \) and \( y_2(t) = t^{1/2} \) are solutions of the differential equation \( yy'' + (y')^2 = 0 \) for \( t > 0 \). Then show that \( y = c_1 + c_2t^{1/2} \) is not, in general, a solution of this equation. Explain why this result does not contradict Theorem 3.2.2.
Proof. Note that \( y_1'(t) = y_1(t)'' = 0 \), \( y_2(t)' = (1/2)t^{-1/2} \), \( y_2'(t) = -(1/4)t^{-3/2} \). Plugging these into the equation we get that these are both solutions to the original equation. For the second part, since \( y' = c_2y_2 \), \( y'' = c_2y_2'' \), we have:

\[
yy'' + (y')^2 = (c_1 + c_2 t^{1/2})(-\frac{c_2}{4}t^{-3/2} + \frac{c_2}{2}t^{-1})^2 = \frac{-c_1 c_2 t^{-3/2} - \frac{c_2^2}{4} t^{-1} + \frac{c_2^2}{4} t^{-1}}{4} = -\frac{c_1 c_2 t^{-3/2}}{4} \neq 0
\]

This result does not contradict the theorem because this equation is not in the form

\[
y'' + p(t)y' + q(t)y = g(t).
\]

\[\blacksquare\]

In Problem 3.2.24 verify that the functions \( y_1 \) and \( y_2 \) are solutions of the given differential equation. Do they constitute a fundamental set of solutions?

**Problem 3.2.24.** \( y'' + 4y = 0; \quad y_1(t) = \cos 2t, \quad y_2(t) = \sin 2t \)

**Proof.** Since

\[
(\cos 2t)'' + 4 \cos 2t = -4 \cos 2t + 4 \cos 2t = 0
\]

and

\[
(\sin 2t)'' + 4 \sin 2t = -4 \sin 2t + 4 \sin 2t = 0
\]

, \( y_1 \) and \( y_2 \) are both solutions of the given equation.

For the second part, by definition we only need to check that \( W(y_1, y_2) \) is nonzero. Since

\[
W(\cos t, \sin t) = \begin{vmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{vmatrix} = \cos^2 2t + \sin^2 2t = 1
\]

, we’re done.

\[\blacksquare\]

\[\triangle\]

**Section 3.3**

\[\triangle\]

In each of Problems 3.3.7 through 3.3.16 find the general solution of the given differential equation.

**Problem 3.3.8.** \( y'' - 2y' + 6y = 0 \)

**Proof.** The characteristic equation

\[
r^2 - 2r + 6 = 0
\]

has roots \( -\frac{-2\pm\sqrt{(-2)^2-4\cdot1\cdot6}}{2} = 1 \pm i\sqrt{5} \). This implies that the equation has a fundamental set of two complex solutions (See Page 151 for what this means) \( \{y_1, y_2\} \) where

\[
y_1(t) = \exp[(1 + i\sqrt{5})t] = e^t(\cos \sqrt{5}t + i \sin \sqrt{5}t)
\]

and

\[
y_2(t) = \exp[(1 - i\sqrt{5})t] = e^t(\cos \sqrt{5}t - i \sin \sqrt{5}t).
\]

It follows that the (real) general solution of the given equation is

\[
y = e^t(c_1 \cos \sqrt{5}t + c_2 \sin \sqrt{5}t).
\]

\[\blacksquare\]
Problem 3.3.9. \( y'' + 2y' - 8y = 0 \)

Proof. The characteristic equation \( r^2 + 2r - 8 = (r + 4)(r - 2) = 0 \) has roots 2, -4. Since the roots are real and different, it follows that the general solution is \( y = c_1 e^{-4t} + c_2 e^{2t} \). ■

Problem 3.3.15. \( y'' + y' + 1.25y = 0 \)

Proof. The characteristic equation \( r^2 + r + 5/4 = 0 \) has roots \(-1/2 \pm i\). Hence the general solution is \( y = e^{-1/2t}(c_1 \cos t + c_2 \sin t) \) ■

In each of Problems 3.3.17 through 3.3.22 find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing \( t \).

Problem 3.3.19. \( y'' - 2y' + 5y = 0, \quad y(\pi/2) = 0, \quad y'(\pi/2) = 2 \)

![Figure 3: Graph 3.3.19.](image)

Proof. The characteristic equation \( r^2 - 2r + 5 = 0 \) has roots \( 1 \pm 2i \). Hence the general solution is \( y = e^t(c_1 \cos 2t + c_2 \sin 2t) \). It has derivative \( y' = 2e^t(-c_1 \sin 2t + c_2 \cos 2t) \). Substituting \( y(\pi/2) = 0, \quad y'(\pi/2) = 2 \), it follows that

\[
\begin{align*}
    e^{\pi/2}(-c_1) &= 0 \\
    2e^{\pi/2}(-c_2) &= 2 
\end{align*}
\]
Therefore \( c_1 = 0, c_2 = -e^{-\pi/2} \), giving the specific solution \( y(t) = -e^{t-\pi/2} \sin 2t \).

Since \( \sin 2t \) is bounded and \(-e^{t-\pi/2}\) goes to \(-\infty\) as \( t \to \infty \), \( y \) goes to \(-\infty\) as \( t \to \infty \).

From the graph we can tell that the function \( y(t) \) has **growing oscillation**.

\[ \square \]

**Problem 3.3.21.** \( y'' + y' + 1.25y = 0, \ y(0) = 3, \ y'(0) = 1 \)

![Figure 4: Graph 3.3.21](image)

**Proof.** From problem 3.3.15, the general solution is \( y = e^{-t/2}(c_1 \cos t + c_2 \sin t) \). Invoking the first initial condition, \( y(0) = 3 \), which implies that \( c_1 = 3 \). Substituting, it follows that \( y = 3e^{-t/2} \cos t + c_2e^{-t/2} \sin t \), and so the first derivative is

\[
y' = -\frac{3}{2}e^{-t/2} \cos t - 3e^{-t/2} \sin t + c_2e^{-t/2} - \frac{c_2}{2}e^{-t/2} \sin t.
\]

Invoking the initial condition, \( y'(0) = 1 \), we find that \(-3/2 + c_2 = 1\), and so \( c_2 = 5/2 \). Hence the specific solution is \( y(t) = 3e^{-t/2} \cos t + \frac{5}{3}e^{-t/2} \sin t \).

Since \( \sin t \) and \( \cos t \) are bounded and \( e^{-t/2} \) goes to 0 as \( t \to \infty \), \( y \) goes to 0 as \( t \to \infty \).

From the graph we can tell that the function \( y(t) \) has **decaying oscillation**.

\[ \square \]

**Problem 3.3.22.** \( y'' + 2y' + 2y = 0, \ y(\pi/4) = 2, \ y'(\pi/4) = -2 \)
Proof. The characteristic equation $r^2 + 2r + 2 = 0$ has roots $-1 \pm i$. Hence the general solution is $y = e^{-t}(c_1 \cos t + c_2 \sin t)$, with derivative $y' = e^{-t}(-(c_1 + c_2) \sin t + (c_2 - c_1) \cos t)$ Substituting $y(\pi/4) = 2, \ y'(\pi/4) = -2$, it follows that

\[
\begin{align*}
\begin{cases} 
  e^{-\pi/4}(\sqrt{2}c_1 + \sqrt{2}c_2) = 2 \\
  e^{-\pi/4} \cdot \sqrt{2}(-2c_1) = -2
\end{cases}
\]

This system of linear equations solve to $c_1 = c_2 = \sqrt{2}e^{\pi/4}$, so the specific solution is

\[
y = \sqrt{2}e^{-t+\pi/4}(\cos t + \sin t)
\]

Since $\sin t$ and $\cos t$ are bounded and $e^{-t}$ goes to 0 as $t \to \infty$, $y$ approaches 0 as $t \to \infty$. From the graph we can tell that the function $y(t)$ has decaying oscillation. 

![Figure 5: Graph 3.3.22](image)

**Problem 3.3.27.** Show that $W(e^{\lambda t} \cos \mu t, e^{\lambda t} \sin \mu t) = \mu e^{2\lambda t}$

Proof.

\[
W(e^{\lambda t} \cos \mu t, e^{\lambda t} \sin \mu t) = \begin{vmatrix}
  e^{\lambda t} \cos \mu t & e^{\lambda t} \sin \mu t \\
  \lambda e^{\lambda t} \cos \mu t - \mu e^{\lambda t} \sin \mu t & \lambda e^{\lambda t} \sin \mu t + \mu e^{\lambda t} \cos \mu t
\end{vmatrix}
\]

\[
= (\lambda e^{2\lambda t} \cos \mu t \sin \mu t + \mu e^{2\lambda t} \cos^2 \mu t)
\]

\[
- (\lambda e^{2\lambda t} \cos \mu t \sin \mu t + \mu e^{2\lambda t} \sin^2 \mu t)
\]

\[
= \mu e^{2\lambda t}(\cos^2 \mu t + \sin^2 \mu t) = \mu e^{2\lambda t}
\]

\[\blacksquare\]
Problem 3.3.29. Using Euler’s formula, show that
\[ \cos t = \frac{e^{it} + e^{-it}}{2}, \quad \sin t = \frac{e^{it} - e^{-it}}{2i} \]

Proof. Recall the beautiful formula of Euler’s:
\[ e^{it} = \cos t + i \sin t \] (1)

Where \( i = \sqrt{-1} \).

Using Euler’s formula and the trig identities \( \cos(-t) = \cos t \), \( \sin(-t) = -\sin t \), we have:
\[ e^{i(-t)} = \cos(-t) + i \sin(-t) = \cos(t) - i \sin(t) \] (2)

Then (1) + (2) gives us the first equation, while (1) – (2) gives us the second. ■

Change of Variables. Sometimes a differential equation with variable coefficients, \( y'' + p(t)y' + q(t)y = 0 \), can be put in a more suitable form for finding a solution by making a change of the independent variable. We explore these ideas in Problems 34 through 46. In particular, in Problem 34 we show that a class of equations known as Euler equations can be transformed into equations with constant coefficients by a simple change of the independent variable. Problems 35 through 42 are examples of this type of equation.

Optional Problem 3.3.34.

Euler Equations. An equation of the form
\[ t^\alpha \frac{d^2 y}{dt^2} + \alpha t \frac{dy}{dt} + \beta y = 0, \quad t > 0, \] (ii)

where \( \alpha \) and \( \beta \) are real constants, is called an Euler equation.

(a) Let \( x = \ln t \) and calculate \( \frac{dy}{dt} \) and \( \frac{d^2 y}{dt^2} \) in terms of \( \frac{dy}{dx} \) and \( \frac{d^2 y}{dx^2} \).

(b) Use the results of part (a) to transform Eq. (ii) into
\[ \frac{d^2 y}{dx^2} + (\alpha - 1) \frac{dy}{dx} + \beta y = 0. \] (iii)

Observe that Eq. (iii) has constant coefficients. If \( y_1(x) \) and \( y_2(x) \) form a fundamental set of solutions of Eq. (iii), then \( y_1(\ln t) \) and \( y_2(\ln t) \) form a fundamental set of solutions of Eq. (ii).

Proof.

(a) \[
\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{dy}{dx} \cdot (\ln t)' = \frac{dy}{dx} \cdot \frac{1}{t},
\]
\[
\frac{d^2 y}{dt^2} = \frac{d}{dt} \left( \frac{dy}{dx} \cdot \frac{1}{t} \right) = \frac{d}{dt} \frac{dy}{dx} \cdot \frac{1}{t} + \frac{dy}{dx} \cdot \frac{d}{dt} \cdot \frac{1}{t} = \frac{d}{dx} \frac{dy}{dx} \cdot \frac{1}{t} + \frac{dy}{dx} \cdot \frac{1}{t} = \frac{d^2 y}{dx^2} \cdot \frac{1}{t^2} - \frac{dy}{dx} \cdot \frac{1}{t^2}
\]
(b) Substituting results of (a) into (ii), we get

\[ t^2 \left( \frac{d^2 y}{dx^2} \cdot \frac{1}{t^2} - \frac{dy}{dx} \cdot \frac{1}{t^2} \right) + \alpha t \left( \frac{dy}{dx} \cdot \frac{1}{t} \right) + \beta y = 0 \]

which immediately implies (iii).

In Problem 3.3.36 use the method of Problem 34 to solve the given equation for \( t > 0 \).

**Optional Problem 3.3.36.** \( t^2 y'' + 4ty' + 2y = 0 \)

**Proof.** Do as in the previous problem, set \( x = \ln t \), then we have

\[ \frac{d^2 y}{dx^2} + (4 - 1) \frac{dy}{dx} + 2y = 0. \]

This ODE has characteristic equation \( r^2 + 3r + 2 = (r + 1)(r + 2) = 0 \), giving the general solution \( c_1 e^{-t} + c_2 e^{-2t} \). So the general solution to the original ODE is \( c_1 t^{-1} + c_2 t^{-2} \).