6.3, 6.4 Step functions and their Laplace Transform

1.

\[ u_c(t) = \begin{cases} 
0 & t < c \\
1 & t \geq c 
\end{cases} \]

functions with discontinuity can be written in terms of step functions.

**Ex 1.**

\[ h(t) = \begin{cases} 
0 & t < 2 \\
1 & 2 \leq t < 3 \\
-1 & t \geq 3 
\end{cases} \]

\[ h(t) = u_2(t) - u_3(t) \]

**Ex 2.**

\[ h(t) = \begin{cases} 
0 & t < \pi \\
\cos(t-\pi) & t \geq \pi 
\end{cases} \]

\[ h(t) = u_{\pi}(t) \cdot \cos(t-\pi) \]

**Ex 3.**

\[ g(t) = \begin{cases} 
0 & t < c \\
f(t-c) & t \geq c 
\end{cases} \]

\[ g(t) = u_c(t) \cdot f(t-c) \]

**Ex 4.**

\[ f(t) = \begin{cases} 
sin t & 0 \leq t < \pi/4 \\
sin t + \cos(t-\pi/4) & t \geq \pi/4 
\end{cases} \]

\[ f(t) = sin t + u_{\pi/4}(t) \cos(t-\pi/4) \]

**2. Laplace Transform of step functions**

\[ L \{ u_c(t) \} \]

\[ = \int_0^\infty e^{-st} u_c(t) \, dt = \int_0^c e^{-st} \, dt = -\frac{1}{s} e^{-st} \bigg|_0^c = \frac{e^{-sc}}{s} \]

\[ L \{ u_c(t) f(t-c) \} \]

\[ = \int_0^\infty e^{-st} u_c(t) f(t-c) \, dt = \int_c^\infty e^{-st} f(t-c) \, dt \]

Let \( u = t-c \)

\[ = \int_0^\infty e^{-su} f(u) \, du \cdot e^{-sc} = e^{-sc} L \{ f(u) \} \]

**Theorem:**

\[ L \{ u_c(t) f(t-c) \} = e^{-sc} L \{ f(t-c) \} \]

\[ u_c(t) L^{-1} \{ F(s) \} = L^{-1} \{ e^{-sc} F(s) \} \]
Ex: find \( \mathcal{L}^{-1}\{e^{-2s} \frac{1}{s^2+1}\} \)

Let \( F(s) = \frac{1}{s^2+1} \), then \( \mathcal{L}^{-1}\{F\} = \sin t \).

So, \( \mathcal{L}^{-1}\{e^{-2s} \frac{1}{s^2+1}\} = \mathcal{L}^{-1}\{e^{-2s} F\} = \mathcal{L}^{-1}\{e^{-2s} \sin t\} \).

(3) Note: Here we have \( t - c \) in \( t \). How about if we have \( s - c \) in \( F \)?

How about \( \mathcal{L}^{-1}\{F(s-c)\} \)?

It turns out that \( e^{ct}f(t) = \mathcal{L}^{-1}\{F(s-c)\} \).

Verify that this is true.

\[
\mathcal{L}\{e^{ct}f(t)\} = \int_0^\infty e^{-st} e^{ct}f(t) \, dt = \int_0^\infty e^{-s(t-c)}f(t) \, dt = \mathcal{L}\{f(t-c)\}
\]

Thus, if \( F(s) = \mathcal{L}\{f(t)\} \), then \( \mathcal{L}\{e^{ct}f(t)\} = F(s-c) \), \( e^{ct}f(t) = \mathcal{L}^{-1}\{F(s-c)\} \).

Ex: find \( \mathcal{L}^{-1}\{\frac{1}{s^2-25+2}\} \)

\[
\frac{1}{s^2-25+2} = \frac{1}{(s-5)^2+1} = F(s-5)
\]

\( \mathcal{L}\{\sin 6t\} = \frac{1}{s^2+1} = F(s) \).

Thus, \( \mathcal{L}^{-1}\{\frac{1}{s^2-25+2}\} = e^5 \sin 5t \).

(4) ODEs with discontinuous forcing

Ex. \( y'' - 2y' + y = g(t) \)

\( y(0) = 0, \ y'(0) = 0 \).

\( g(t) = \begin{cases} 0 & 0 \leq t < 1 \text{ or } t \geq 2 \\ 1 & 1 \leq t < 2 \end{cases} \)

\[
\mathcal{L}\{y'' - 2y' + y\} = s^2 \mathcal{L}\{y\} - sy(0) - y'(0) - 2s \mathcal{L}\{y\} + 2 \mathcal{L}\{y\} + 2 \mathcal{L}\{y\}
\]

\[
= (s^2 - 25 + 2) \mathcal{L}\{y\}
\]

\[
\mathcal{L}\{g(t)\} = \mathcal{L}\{U_1(t) - U_2(t)\} = \frac{e^{-5}}{5} - \frac{e^{-25}}{5}
\]
\[ (s^3 - 25s^2 + 1) L(y) = \frac{e^{-5} - e^{-10}}{s} \]

\[ L(y) = \frac{1}{s(s^3 - 25s + 1)} (e^{-5} - e^{-10}) \]

\[ y = L^{-1} \left( \frac{1}{s(s^3 - 25s + 1)} (e^{-5} - e^{-10}) \right) \]

\[ y = u_1(t)f(t-1) + u_2(t)f(t-2) \]

Where \[ L(f(t)) = \frac{1}{s(s^3 - 25s^2 + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 - 5s + 2} \]

\[ \Rightarrow \quad AS^2 - 2s + 2A + BS^2 + Cs = 1 \]

\[ \begin{cases} 
A + B = 0 \\
2A + C = 0 \\
2A = 1 
\end{cases} \]

\[ A = \frac{1}{2}, \quad B = -\frac{1}{2}, \quad C = 1 \]

\[ \frac{1}{s(s^3 - 25s + 1)} = \frac{1}{2s} - \frac{1}{2} \frac{(s-1)}{(s-1)^2 + 1} + \frac{1}{2} \frac{1}{(s-1)^2 + 1} \]

From Table 6.2.1,

\[ L(1) = \frac{1}{s^2 + 1}, \quad L(s \sin t) = \frac{1}{s^2 + 1}, \quad L(s \cos t) = \frac{1}{s^2 + 1} \]

So,

\[ L^{-1} (\frac{1}{s}) = 1, \quad L^{-1} (\frac{1}{s^2 + 1}) = e^{t \cos t}, \quad L^{-1} (\frac{1}{(s-1)^2 + 1}) = e^{t \sin t}. \]

\[ \Rightarrow \quad L^{-1} \left( \frac{1}{s(s^3 - 25s + 1)} \right) = \frac{1}{2} - \frac{1}{2} e^{t \cos t} + \frac{1}{2} e^{t \sin t}. \]

Hence,

\[ y = u_1(t) \left( \frac{1}{2} - \frac{1}{2} e^{t \cos (t-1)} + \frac{1}{2} e^{t \sin (t-1)} \right) \]

\[ + u_2(t) \left( \frac{1}{2} - \frac{1}{2} e^{t \cos (t-2)} + \frac{1}{2} e^{t \sin (t-2)} \right) \]