Please answer the following questions. Because this test is open book and open note, you will not get credit for answers unless you demonstrate how you arrived at them. In short, please show all work.

Problem 1.

Please find the specific solution to the transport equation:

$$\frac{1}{2 + \sin(x)} \partial_x u + (y + 1) \partial_y u = 0,$$

with the initial data at $y = 0$:

$$u(x, 0) = \cos(x) - 2x.$$
Problem 2.

For this problem, let $u(x, t)$ be the specific solution to the wave equation:

$$
\begin{align*}
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2}, & \text{on} & \quad -\infty < x < \infty \\
u(x, 0) &= 0, \\
u_t(x, 0) &= e^{-x^2}.
\end{align*}
$$

Please answer the following:

a) Compute the energy at time $t = 100$ for this solution $E(100)$. Please explain your answer carefully.

b) Is it true that the solution $u(x, t)$ is always strictly positive for $0 < t$? That is, is it true that $0 < u(x, t)$ for all $0 < t$. 

Problem 3.

Please answer the following:

a) Suppose that one has two solutions $u(x, t)$ and $v(x, t)$ to the heat equation with Dirichlet boundary conditions on the interval $[0, \pi]$:

$$u_t = u_{xx},$$
$$u(0, t) = u(\pi, t) = 0,$$

and:

$$v_t = v_{xx},$$
$$v(0, t) = v(\pi, t) = 0.$$

Suppose that at the initial time $t = 0$ one has the inequality:

$$v(x, 0) \leq u(x, 0).$$

Show that for all positive times $0 < t$ one has that:

$$v(x, t) \leq u(x, t).$$

Please explain carefully your answer. (Hint: Consider the function $w = u - v$.)

b) Notice that the function:

$$u(x, t) = e^{-t} \sin(x),$$

solves the above heat equation with zero Dirichlet boundary conditions on the interval $[0, \pi]$. Show that if $v(x, t)$ is any solution to the (same) heat equation boundary value problem:

$$v_t = v_{xx},$$
$$v(0, t) = v(\pi, t) = 0,$$

with the additional property that at $t = 0$ (on $[0, \pi]$):

$$-\sin(x) \leq v(x, t) \leq \sin(x),$$

then one always has:

$$|v(x, t)| \leq e^{-t}.$$  

(Hint: Recall that if $-u \leq v \leq u$, with $0 \leq u$, then one also has $|v| \leq u$.)