1. Let \( u, v \) be eigenfunctions of \(-\Delta\) on a bounded domain \( D \subset \mathbb{R}^3 \) satisfying a Robin boundary condition. This means we have \(-\Delta u = \lambda u, -\Delta v = \mu v\) and

\[
\frac{\partial w}{\partial n} + a(\mathbf{x})w(\mathbf{x}) = 0 \quad \text{on} \quad \partial D, \quad \text{where} \quad w = u, v.
\]

(a) Show that the eigenvalue \( \lambda \geq 0 \). Hint: Show that \( \lambda(u, u) = (-\Delta u, u) \geq 0 \) using a Green’s identity. Here, as usual

\[
(f, g) = \int \int_D f(\mathbf{x}) g(\mathbf{x}) \, d\mathbf{x}
\]

for any two functions \( f, g \) on \( D \).

(b) Show that \( \int \int_D u(\mathbf{x}) v(\mathbf{x}) \, d\mathbf{x} = 0 \) if \( \lambda \neq \mu \). Hint: Show that \( \lambda(u, v) = \mu(u, v) \), using a Green’s identity.

2. Let \( D \) be the disk \( x^2 + y^2 \leq a^2 \). We have shown in class that the eigenfunctions \( v \) of \(-\Delta\) satisfying \(-\Delta v = \lambda v\) are given by

\[
v(r, \theta) = J_n(\sqrt{\lambda}r)(A_n \cos n\theta + B_n \sin n\theta)
\]

(a) What are the eigenfunctions which do not depend on \( \theta \) and which satisfy the boundary condition \( \partial v / \partial n = 0 \)? What are their eigenvalues? (It is OK to describe the eigenvalues as the zeros of one or several functions; it may not be possible to give a more explicit description).

(b) Let \( \lambda_n \) be the eigenvalues of (a) with eigenfunctions \( v_n \). One can show that the function \( f(r) = r^2 \) can be written as

\[
r^2 = \sum_{n=1}^{\infty} A_n v_n.
\]

How does one calculate the coefficients \( A_n \)? It is OK to write down formulas for \( A_n \) involving integrals without calculating them.