Solutions to Problems from Chapters 7.1 and 7.2

Problem # 3 Section 7.1. To prove this, we simply use the energy identity, which is equation (7) on p. 173 of the text. From this we see that:

\[
\iiint_{\Omega} |\nabla u|^2 \, d\vec{x} = \iint_{\partial\Omega} u \frac{\partial u}{\partial n} \, dS,
\]

\[
= \iint_{\partial\Omega} -a \cdot u^2 \, dS.
\]

The second line follows simply from using the boundary condition \( \frac{\partial u}{\partial n} + a \cdot u = 0 \). Recall that we have the point-wise condition \( 0 < a(\vec{x}) \), for every point \( \vec{x} \in \partial\Omega \). Therefore, unless we have \( u^2 \equiv 0 \) on \( \partial\Omega \), we would have:

\[
(1) \quad \iiint_{\Omega} |\nabla u|^2 \, d\vec{x} = \{\text{Something strictly negative}\},
\]

which is clearly impossible. Therefore, it must actually be that \( u^2 \equiv 0 \) on \( \partial\Omega \), which is to say:

\[
u|_{\partial\Omega} = 0.
\]

By the usual uniqueness for the Dirichlet problem, we then have that \( u \equiv 0 \) in all of \( \Omega \).

Problem # 2 Section 7.2. As indicated in the hint, we use Green’s second identity for the region \( D \), which is a ball of radius \( R \) around 0 from which a small ball of radius \( \epsilon \) is cut out; here \( R \) is chosen large enough so that \( \phi \) and its gradient are zero on the sphere of radius \( R \). Moreover \( u = 1/|x| \) and \( v = \phi \). Then, as \( v(x) = 0 \) and \( \partial v/\partial n(x) = \nabla \phi \cdot (x/|x|) = 0 \) for \( |x| = R \), we get from Green’s second identity

\[
(2) \quad \iiint_D \frac{1}{|x|^3} \Delta \phi(x) - 0 \, dx = \iint_{|x| = \epsilon} \frac{1}{|x|} \partial \phi/\partial n - \phi \partial \partial (\frac{1}{|x|}) \, dS;
\]

here we used the fact that \( \phi \) and its gradient are zero on the sphere of radius \( R \), hence we only need to consider the boundary coming from the small ball. Exactly as in the book on pages 186/187, we use the fact that \( \partial/\partial n = -\partial/\partial r \), as the normal vector on the small sphere points towards the origin; moreover, we have \( |x| = r \). We can now rewrite our formula above as

\[
(3) \quad \iiint_{\mathbb{R}^3} \frac{1}{|x|} \Delta \phi(x) \, dx = \iint_{r=\epsilon} -\frac{1}{r} \partial \phi/\partial r + \phi \frac{\partial}{\partial r} (\frac{1}{r}) \, dS;
\]

in view of our conditions on \( \phi \) extending the integration on the left hand side to all of \( \mathbb{R}^3 \) does not change the value of the integral. It only remains to show now that the right hand side goes to \(-4\pi u(0)\) if \( \epsilon \to 0 \). This was done in class, and in the book on page 167. Alternatively, using spherical coordinate, we can rewrite the right hand side as

\[
(4) \quad -\int_0^\pi \int_0^{2\pi} \left[ \frac{1}{\epsilon} \frac{\partial \phi}{\partial r} + \phi \frac{1}{\epsilon^2} \right] \epsilon^2 \sin \theta \, d\phi d\theta = -\int_0^\pi \int_0^{2\pi} \left[ \epsilon \frac{\partial \phi}{\partial r} + \phi(\epsilon, \varphi, \theta) \right] \sin \theta \, d\phi d\theta.
\]

You should check yourself that the right hand side goes to \(-4\pi u(0)\) if \( \epsilon \to 0 \). You can use that \( \partial \phi/\partial r \) is continuous, and hence its absolute value bounded by a constant on the ball of radius \( \epsilon \).