Here we collect a number of useful properties of Bessel functions. Using its power series expansion
\[ J_s(z) = \sum_{j=0}^{\infty} \frac{1}{\Gamma(j+1)\Gamma(j+s+1)} \left( \frac{z^2}{2} \right)^j, \]
one can prove (see homework) that
\[ J_{s\pm 1}(z) = \frac{s}{z} J_s(z) \mp J'_s(z). \tag{1} \]

We have shown the following result in class.

**Lemma 1** With the definitions above, we have
\[(a) \quad \int_0^a J_s(z)^2 z \, dz = \frac{1}{2} [a^2 J'_s(a)^2 + (a^2 - s^2) J_s(a)^2], \]
\[(b) \quad \text{If } J_s(\beta a) = 0, \text{ then } \int_0^a J_s(\beta r)^2 r \, dr = \frac{1}{2} a^2 J'_s(\beta a)^2 = a^2 J_{s\pm 1}(\beta a)^2, \]
\[(c) \quad (\tilde{u}_{n,m}, \tilde{u}_{n,m}) = \frac{\pi}{2} a^2 J_{s\pm 1}(\sqrt{\lambda_{n,m}} a)^2. \]

**Proof.** For the proofs of (a) and (b), see p 284, *Normalizing Constants* in Chapter 10.5 of our book. For part (c) we use the definition of \( \tilde{u}_{n,m} \) and of the inner product \( (\tilde{u}_{n,m}, \tilde{u}_{n,m}) \), see the proof of the first theorem in the previous notes on vibrations of a drumhead. It then suffices to apply part (b) to the integral, with \( \beta = \sqrt{\lambda_{n,m}}. \)

We shall also need Bessel functions \( J_s \) with \( s = n + \frac{1}{2} \) a half-integer. Using the substitution \( u = z^{-1/2} v \), it is shown in a homework problem that Bessel’s equation transforms to
\[ v'' + \left( 1 - \frac{s^2 - \frac{1}{4}}{z^2} \right) v = 0. \]
It follows that for \( s = \frac{1}{2} \) we get the differential equation \( v'' + v = 0 \). As we also want a finite value for \( u(z) = z^{-1/2} v(z) \) for \( z \to 0 \), we obtain the solution
\[ J_{1/2}(z) = \frac{\sqrt{2}}{\pi z} \sin z, \]
where the factor \( \sqrt{2} / \pi \) is added for making certain other formulas nicer later on.

**Lemma 2** The half-integer Bessel functions \( J_{n+\frac{1}{2}} \) are given by the formula
\[ J_{n+\frac{1}{2}}(z) = ( -1 )^n \sqrt{\frac{2}{\pi}} z^{n+\frac{1}{2}} \left( \frac{1}{z} \frac{d}{dz} \right)^n \frac{\sin z}{z}. \]

**Proof.** The proof goes by induction on \( n \). It is easily checked for \( n = 0 \). By induction assumption for \( n - 1 \), we can write
\[ \left( \frac{1}{z} \frac{d}{dz} \right)^{n-1} \frac{\sin z}{z} = ( -1 )^{n-1} \frac{1}{z^{n}} \sqrt{\frac{\pi}{2}} J_{n-\frac{1}{2}}(z). \]
Using this, we can reduce differentiating $n$ times to only differentiating once in the formula below:

\[
(-1)^n \sqrt{\frac{2}{\pi}} z^{n+\frac{1}{2}} \left( -1 \frac{d}{dz} \right)^n \sin z = -z^{n-\frac{1}{2}} \frac{d}{d_z} (J_{n-\frac{1}{2}}(z) z^{\frac{1}{2} - n}) =
\]

\[
= -z^{n-\frac{1}{2}} [J'_{n-\frac{1}{2}}(z) z^{\frac{1}{2} - n} + (\frac{1}{2} - n) z^{-\frac{1}{2} - n} J_{n-\frac{1}{2}}(z) ] =
\]

\[
= -[J'_{n-\frac{1}{2}}(z) - \frac{n-\frac{1}{2}}{z} J_{n-\frac{1}{2}}(z)] = J_{n+\frac{1}{2}}(z),
\]

where we used the recursive formula (1) for $s = n - \frac{1}{2}$. This finishes the proof.

**Example** Check that I did not make a mistake by calculating

\[
J_{3/2} = \sqrt{\frac{2}{\pi}} z^{-3/2} (\sin z - z \cos z).
\]

What about its limit if $z \to 0$? Recall that it should be finite!