We define the upper and lower integrals by Definition of integral here

\[
U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \quad \text{and} \quad L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1});
\]

where \( m_i \) and \( M_i \) are infimum and supremum of \( f \) on the interval \([x_{i-1}, x_i]\).

**Definition of integral** We define the upper and lower integrals by

\[
\int_a^b f = \inf U(f, P), \quad \int_a^b f = \sup L(f, P),
\]

where the inf and sup are taken over all partitions of \([a, b]\). A bounded function is called integrable if upper and lower integrals coincide.

**Theorem** (a) The lower integral of \( f \) is always less or equal than the upper integral.

(b) (Archimedes-Riemann) A bounded function \( f : [a, b] \to \mathbb{R} \) is integrable if and only if there exists a sequence of partitions \( (P_n) \) such that

\[
\lim_{n \to \infty} |U(f, P_n) - L(f, P_n)| = 0.
\]

**Continuous, integrable and differentiable functions** Recall that \( f : [a, b] \to \mathbb{R} \) is called continuous at \( x_0 \) if for every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that \( |f(x) - f(x_0)| < \epsilon \) if \( |x - y| < \delta \).

**Theorem** (a) Every continuous function is integrable

(b) Assume \( f : [a, b] \to \mathbb{R} \) is continuous and \( f(x) \geq 0 \) for all \( x \in [a, b] \). If there exists \( x_0 \in [a, b] \) such that \( f(x_0) > 0 \), then \( \int_a^b f > 0 \)

We have the implications \( f \) differentiable \( \Rightarrow \) \( f \) continuous \( \Rightarrow \) \( f \) integrable.

**Fundamental Theorems of Calculus** (a) Assume that \( F \) is differentiable on \((a, b)\) and continuous on \([a, b]\) such that also \( F'(x) \) is continuous on \([a, b]\). Then

\[
\int_a^b F'(x) = F(b) - F(a).
\]

(b) Assume \( f : [a, b] \to \mathbb{R} \) is continuous. Then

\[
\frac{d}{dx} \left[ \int_a^x f \right] = f(x), \quad \frac{d}{dx} \left[ \int_x^b f \right] = -f(x).
\]

**Taylor polynomials and approximations** Let \( I \) be an open interval and let \( f : I \to \mathbb{R} \) be a function with \( n \) derivatives. Then its \( n \)-th Taylor polynomial \( p_n \) at \( x_0 \in I \) is defined to be

\[
p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k.
\]

The function \( f \) is given by its Taylor series at \( x \), i.e. \( f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k \), if \( f(x) = \lim_{n \to \infty} p_n(x) \).

**Lagrange Remainder Theorem** Assume \( f : I \to \mathbb{R} \) has \( n + 1 \) derivatives. Let \( x_0, x \in I \). Then there exists a number \( c \) between \( x_0 \) and \( x \) such that

\[
f(x) - p_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}.
\]
Lemma (a) Let $c$ be a constant. Then we have
\[
\lim_{n \to \infty} \frac{c^n}{n!} = 0.
\]
(b) Let $(c_n)$ be a sequence such that $\lim_{n \to \infty} \frac{|c_{n+1}|}{|c_n|} = r$.
   (i) If $r < 1$, then $\lim_{n \to \infty} c_n = 0$
   (ii) If $r > 1$, then $(c_n)$ is an unbounded sequence.

Weierstrass Approximation Theorem Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and let $\epsilon > 0$. Then there exists a polynomial $p$ such that $|p(x) - f(x)| < \epsilon$ for all $x \in [a, b]$.

Pointwise and uniform convergence Let $f_n : D \to \mathbb{R}$ be a sequence of functions, and let $f : D \to \mathbb{R}$.
   (a) The sequence $(f_n)$ converges to $f$ pointwise if $\lim f_n(x) = f(x)$ for all $x \in [a, b]$.
   (b) The sequence $(f_n)$ converges to $f$ uniformly if for every $\epsilon > 0$ we can find an $N$ such that $|f_n(x) - f(x)| < \epsilon$ for all $x \in [a, b]$ and all $n \geq N$.

Theorem Assume $f_n \to f$ uniformly, and $D = [a, b]$.
   (a) If all $f_n$'s are continuous, then so is $f$.
   (b) If all $f_n$'s are integrable, then so is $f$. Moreover, in this case $\lim_{n \to \infty} \int_a^b f_n = \int_a^b f$.
   (c) Assume all $f_n$'s are differentiable. If the $f'_n$'s converge uniformly to a function $g$, and the functions $f_n$ converge pointwise to the function $f$, then $f$ is differentiable and $f' = g = \lim_{n \to \infty} f'_n$.

Theorem Assume for some $r > 0$ the function $f : (-r, r) \to \mathbb{R}$ is given by the power series
\[
f(x) = \sum_{k=0}^{\infty} c_k x^k, \quad \text{if } |x| < r.
\]
Then $f$ has derivatives of all orders. In particular
\[
f'(c) = \sum_{k=0}^{\infty} k c_k x^{k-1}, \quad \text{if } |x| < r.
\]