Math 181A Worksheet Week 2

Examples:

1. Method of Moments
   Let our sample be 6, 5, 8 from $Binom(10, p)$. In your distribution sheet $EX = \mu = np$ if $X \sim Binom(n, p)$. Find the Method of Moments estimator for $p$ and compute its estimate.

Solution:
   In the given setup, the first element of Binomial parameter is known as $n = 10$. There is only one unknown parameter $p$. By the Method of Moments, we shall solve $p$ from the first moment. As
   \[ EX = 10p \]
   so
   \[ p = \frac{EX}{10} \]
   Replace the population moment $EX$ by sample moment $\frac{X_1+X_2+X_3}{3}$, we have the Method of Moments estimator
   \[ \hat{p}(X_1, X_2, X_3) = \frac{X_1+X_2+X_3}{3} = \frac{X_1 + X_2 + X_3}{30} \]
   Plug-in the observed values for $X_1$, $X_2$ and $X_3$, we have its estimate
   \[ \hat{p}(6, 5, 8) = \frac{6 + 5 + 8}{30} = \frac{19}{30} \]

2. Maximum Likelihood
   Let our sample be 1.44, 6.08 from $Exp(\theta)$. In your distribution sheet, the density is $f_\theta(x) = \frac{1}{\theta} e^{-x/\theta}$. Find the Maximum Likelihood estimator for $\theta$ and compute its estimate.

Solution:
The likelihood function (joint density) with 2 observations is
   \[ L(\theta; x_1, x_2) = f_\theta(x_1)f_\theta(x_2) = \frac{1}{\theta} e^{-x_1/\theta} \frac{1}{\theta} e^{-x_2/\theta} = \frac{1}{\theta^2} e^{-(x_1+x_2)/\theta} \]
   Recall that density should be positive at observed values (otherwise the model is ill-chosen) and the ln function is strictly monotone increasing. Like the example in the book, compute log-likelihood to turn product into sum:
   \[ l(\theta) = \ln(L(\theta)) = -2 \ln \theta - (x_1 + x_2)/\theta \]
Next, solve the optimization (maximization) problem with the method from 20A.
Find critical points through differentiation:

\[ l'(\theta) = -\frac{2}{\theta} + \frac{x_1 + x_2}{\theta^2} = 0 \]

So \( \theta_c = \frac{x_1 + x_2}{2} \).
Then compare \( l(\theta_c) \) with the boundary values/limits \( l(0+) \) and \( l(+\infty) \). Note \( l(\theta_c) \) is finite while \( l(0+) = l(+\infty) = -\infty \), so the global maximal is achieved at \( \theta_c \).
The ML estimator is therefore

\[ \hat{\theta}(X_1, X_2) = \frac{X_1 + X_2}{2} \]

and its estimate is

\[ \hat{\theta}(1.44, 6.08) = \frac{1.44 + 6.08}{2} = 3.76 \]

Practice:

1. Suppose \( X_1, \ldots, X_6 \overset{iid}{\sim} \text{Poisson}(\lambda) \). In your distribution sheet \( \mathbb{E}X = \mu = \lambda \) if \( X \sim \text{Poisson}(\lambda) \). Find the Method of Moments estimator for \( \lambda \).

**Solution:** Solve parameter from moment:

\[ \mathbb{E}X = \lambda \iff \lambda = \mathbb{E}X \]

Give the estimator:

\[ \hat{\lambda} = \frac{1}{6} (X_1 + X_2 + X_3 + X_4 + X_5 + X_6) \]

2. Let \( Y \sim \mathcal{N}(1, \sigma^2) \) and \( Y_1, Y_2, Y_3 \) be its i.i.d. sample.

- In your distribution sheet \( \mathbb{E}X = \mu \) if \( X \sim \mathcal{N}(\mu, \sigma^2) \). Does that help deriving a Method of Moments estimator for \( \sigma^2 \)?
- In your distribution sheet \( \text{Var}(X) = \sigma^2 \) if \( X \sim \mathcal{N}(\mu, \sigma^2) \). Compute \( \mathbb{E}Y^2 \).
- Give an estimator for \( \sigma^2 \) by the idea of Method of Moments.

**Solution:**

\( \mathbb{E}Y = 1 \), so it is impossible to solve \( \sigma^2 \) from this equation. Thus, Method of Moments fails.

\[ \mathbb{E}Y^2 = \text{Var}Y + (\mathbb{E}Y)^2 = \sigma^2 + 1 \]
We may solve $\sigma^2 = \mathbb{E}Y^2 - 1$. That suggests a plug-in estimator:

$$\hat{\sigma}^2(Y_1, Y_2, Y_3) = \frac{1}{3}(Y_1^2 + Y_2^2 + Y_3^2) - 1$$

3. Suppose $1, 1 \overset{iid}{\sim} \text{Bernoulli}(p)$. The p.m.f. is $f(x) = p^x(1 - p)^{1-x}, x \in \{0, 1\}$ for $\text{Bernoulli}(p)$. Find the Maximum Likelihood estimate for $p$.

**Solution:**

The likelihood is

$$L(p; 1, 1) = p^1(1 - p)^0 p^1(1 - p)^0 = p^2$$

This is a strictly monotone increasing function of $p$. Thus, $p = 1$ maximizes the likelihood, i.e. ML estimate $\hat{p}(1, 1) = 1$.

**Remark:** The critical point is not the maximal in this case! You must check the boundary values.