Math 181B Worksheet Week 2

Homework Hints:

1. Variance Covariance Matrix for a random vector \( \mathbf{Y}_n \): 
   \[ \Sigma_n = \mathbb{E} \mathbf{Y} \mathbf{Y}^T - \mathbb{E} \mathbf{Y} (\mathbb{E} \mathbf{Y})^T. \]
   You need to use i.i.d. r.v. \( X_1, \ldots, X_n \) take values in \( 1, \ldots, k \) to compute the expectations.
   \[ \mathbb{E} Y_i Y_j = \mathbb{E} \left[ \sum_{s=1}^{n} 1\{X_s = i\} \sum_{s=1}^{n} 1\{X_s = j\} \right] \]

2. Prove Maximum Likelihood \( \sup L(\theta) \) is invariant under invertible transformation.
   Then, view the family of distribution as following:
   \[ p_1 = 1 - e^{-e^{\mu_1}}, p_2 = e^{-e^{\mu_1}} - e^{-e^{\mu_1} - e^{\mu_2}}, p_3 = e^{-e^{\mu_1} - e^{\mu_2}}, (\mu_1, \mu_2) \in \mathbb{R}^2 \]
   Then, it is a typical nested test \( H_0 : \mu_1 = \mu_2 \) versus \( H_1 : \mu_1 \neq \mu_2 \). Also, learn from the sample code to do part (c).

3. Prove the following statement: \( \lambda(X) \) is a monotone decreasing function of \( |2X - n| \) if and only if
   (a) it is symmetric at \( 2X = n \); id est \( \lambda(X) = \lambda(n - X) \),
   (b) \( \lambda(X) \) is monotone decreasing in \( X \) when \( 2X > n \).

4. Similar to problem 3 in worksheet 1.
   (a) While doing MLE, you need to solve both \( \hat{\mu} \) and \( \hat{\sigma}^2 \). The region where \( LR \geq 1 \)
   is \( \hat{\sigma}^2 \leq \sigma_0^2 \). Show under null sup \( \sigma^2 \leq \sigma_0^2 \) \[ \mathbb{P}(\sigma^2 \leq \sigma_0^2) = 1 \]
   so include \( LR \geq 1 \) in your rejecting region will make your type 1 error 100%.
   (b) Then, \( LR \) is a increasing function of the given \( T \) when \( \hat{\sigma}^2 > \sigma_0^2 \). Remember the identity
   \[ \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2 \]
   (c) You may want to write the test statistic as
   \[ T = \frac{\sigma^2}{\sigma_0^2} \sum_{i=1}^{n} (X_i - \bar{X})^2 \]
   Recognize a \( \chi^2 \) distribution and notice picking \( \sigma^2 = \sigma_0^2 \in \Theta_0 \) would maximize the rejection rate.

Practice for NeymanPearson Lemma:

\( X_1, \ldots, X_n \overset{iid}{\sim} \text{Exp}(\theta) \),

Find the Most Powerful \( \alpha = 0.05 \) test for \( H_0 : \theta = 1 \) versus \( H_1 : \theta = 2 \).