Math 181B Worksheet Week 2 Solution

Practice: Test the Following Hypothesis by GLRT
You may write your answer in quantile.

1. Simple vs Many: \( X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, 1) \). At \( \alpha \) level,

\[
L(\mu) = (2\pi)^{-n/2} \exp \left( -\frac{1}{2} \sum_{i=1}^{n} (X_i - \mu)^2 \right)
\]

(a) \( H_0 : \mu = 0 \) versus \( H_1 : \mu \neq 0 \).

Solution: Two sided simple vs many

MLE: \( \hat{\mu} = \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \).

GLR = \( \frac{L(0)}{L(\hat{\mu})} = \exp \left( -\frac{1}{2} \sum_{i=1}^{n} X_i^2 + \frac{1}{2} \sum_{i=1}^{n} (X_i - \bar{X})^2 \right) = \exp \left( -\frac{n}{2} \bar{X}^2 \right) \)

Easier Solution (asymptotic): The null parameter space is dimension 0. The full parameter space (union of null and alternative) is dimension 1. By the GLR theorem for two sided test,

\[
-2 \log(GLR) = n\bar{X} \xrightarrow{F} \chi_1^2 = \chi_1^2
\]

Reject \( H_0 \) if \( n\bar{X} > \chi_{1,\alpha}^2 \), where \( \chi_{1,\alpha}^2 \) is the \( 1-\alpha \) quantile of \( \chi_1^2 \) distribution.

Harder Solution (exact): Recognize GLR is the monotone decreasing function of \( n\bar{X}^2 \). The test \( GLR \leq C \) is equivalent to \( n\bar{X}^2 \geq -2 \log C = C' \). Under \( H_0 \),

\[
\bar{X} \sim N(0, 1/n) \Rightarrow \sqrt{n} \bar{X} \sim N(0, 1) \Rightarrow n\bar{X}^2 \sim \chi_1^2
\]

Reject \( H_0 \) if \( n\bar{X} > \chi_{1,\alpha}^2 \), where \( \chi_{1,\alpha}^2 \) is the \( 1-\alpha \) quantile of \( \chi_1^2 \) distribution.

(b) \( H_0 : \mu = 0 \) versus \( H_1 : \mu > 0 \).

Solution: One sided simple vs many

Pick arbitrary \( \mu_1 > 0 \) from alternative \( H_1 \). Do the simple vs simple test \( H_0 : \mu = 0 \) versus \( H_1 : \mu = \mu_1 \).

\[
LR = \frac{L(0)}{L(\mu_1)} = \exp \left( -\frac{1}{2} \sum_{i=1}^{n} X_i^2 + \frac{1}{2} \sum_{i=1}^{n} (X_i - \mu_1)^2 \right) = \exp \left( -n\bar{X}\mu_1 + n\mu_1^2/2 \right)
\]

Notice \( \mu_1 > 0 \), so \( LR \) is a monotone decreasing function of \( \bar{X} \). The LR test \( LR \leq C \) is equivalent to the test \( \bar{X} > C' \) with proper choice of constants. Under \( H_0 \),

\[
\bar{X} \sim N(0, 1/n) \Rightarrow Z = \sqrt{n} \bar{X} \sim N(0, 1)
\]
Reject $H_0$ if $Z > Z_\alpha$, where $Z_\alpha$ is the $1 - \alpha$ quantile of standard Normal distribution. Notice the resulting test does not depend on the choice of $\mu_1$. Therefore, it is also the LR test for the composite alternative $H_1 : \mu > 0$.

(c) (Many vs Many) $H_0 : \mu \leq 0$ versus $H_1 : \mu > 0$.

Solution:

Easier Solution: By a result from lecture, reduce one-sided composite null $H_0 : \mu \leq 0$ to simple null $H_0 : \mu = 0$.

Harder Solution: Use GLR test. MLE $\hat{\mu} = \bar{X}$.

MLE under null is now a constrained maximization problem. Notice the log-likelihood

$$l(\mu) = \log L(\mu) = -n/2 \log(2\pi) - \frac{1}{2} \left( n\mu^2 - 2n\mu \bar{X} + \sum_{i=1}^{n} X_i^2 \right)$$

is a concave function. In fact, it is a quadratic function. By its derivative,

$$l'(\mu) = -n\mu + n\bar{X} = \begin{cases} > 0 & l(\mu) \text{ increasing if } \mu < \bar{X} \\ < 0 & l(\mu) \text{ decreasing if } \mu > \bar{X} \end{cases}$$

Depending on the data, the MLE under null has two cases

$$\hat{\mu}_0 = \begin{cases} 0 & \text{if } 0 < \bar{X} \\ \bar{X} & \text{if } 0 \geq \bar{X} \end{cases}$$

Basically, we need to discuss the LR under the two cases.

Case 1: If we observe $0 \geq \bar{X}$, then $\hat{\mu}_0 = \hat{\mu} = \bar{X}$. GLR $= L(\hat{\mu}_0)/L(\hat{\mu}) = L(\bar{X})/L(\bar{X}) = 1$. Rejecting GLR $= 1$ means that we must reject all $GLR \leq 1$ since the GLR test has the form $GLR \leq C$. Since $GLR \leq 1$ with probability 1, so this means the type 1 error is always 100%, which is contradictory to the given level $\alpha$. Therefore, we must always accept $H_0$ if $\hat{\mu}$ is contained in null hypothesis.

Case 2: 1: If we observe $0 < \bar{X}$, then $\hat{\mu}_0 = 0$.

$$GLR = \exp \left( -\frac{1}{2} \sum_{i=1}^{n} X_i^2 + \frac{1}{2n} \sum_{i=1}^{n} (X_i - \bar{X})^2 \right) = \exp \left( -n\bar{X}^2/2 \right)$$

Like a) part, we identify GLR is a decreasing function of $n\bar{X}^2$. Or, GLR is a decreasing function of $\bar{X}$ if $0 < \bar{X}$.

As a conclusion, reject $H_0$ if $0 < \bar{X}$ and $n\bar{X}^2 > \chi^2_{1,\alpha}$ (or in short, $\sqrt{n}\bar{X} > Z_\alpha$).

2. Many vs Many: $X_1, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$. At $\alpha$ level,

$$L(\mu, \sigma^2) = (\sigma^2 2\pi)^{-n/2} \exp \left( -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2 \right)$$
(a)  $H_0: \mu = 0$ versus $H_1: \mu \neq 0$.

Solution:

MLE: $\hat{\mu} = \bar{X} = 1/n \sum_{i=1}^{n} X_i$, $\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2$.

MLE under null: $\sigma^2_0 = \frac{1}{n} \sum_{i=1}^{n} X_i^2$.

$$GLR = \frac{L(0, \sigma^2_0)}{L(\hat{\mu}, \hat{\sigma}^2)} = \left(\frac{\sigma^2_0}{\hat{\sigma}^2}\right)^{-\frac{n}{2}} \exp \left(-\frac{1}{2\hat{\sigma}^2} \sum_{i=1}^{n} X_i^2 + \frac{1}{2\sigma^2_0} \sum_{i=1}^{n} (X_i - \bar{X})^2\right) = \left(\frac{\sigma^2_0}{\hat{\sigma}^2}\right)^{-\frac{n}{2}}$$

Easier Solution (asymptotic): The null parameter space is dimension 1. The full parameter space (union of null and alternative) is dimension 2. By the GLR theorem for two sided test, and they are independent.

$$T = -2\log(GLR) = n \log \left(\frac{\sigma^2_0}{\hat{\sigma}^2}\right) \xrightarrow{\mathcal{L}} \chi^2_{2-1} = \chi^2_1$$

Reject $H_0$ if $T > \chi^2_{1, \alpha}$, where $\chi^2_{1, \alpha}$ is the $1 - \alpha$ quantile of $\chi^2_1$ distribution.

Harder Solution (exact): Recognize GLR is the monotone decreasing function of $\sigma^2_0/\hat{\sigma}^2$. The test $GLR \leq C$ is equivalent to $\sigma^2_0/\hat{\sigma}^2 \geq C^{-2/n} = C'$. Under $H_0$, $n\bar{X}^2/\sigma^2 \sim \chi^2_1$, $\sum_{i=1}^{n} (X_i - \bar{X})^2/\sigma^2 \sim \chi^2_{n-1}$ and they are independent. Thus, $F = (n-1) \cdot (\hat{\sigma}^2_0/\hat{\sigma}^2 - 1) \sim F_{1,n-1}$. $F$ is a monotone increasing transformation of $\sigma^2_0/\hat{\sigma}^2$, so the LR test is equivalent to $F \geq (n-1) \cdot (C' - 1) = C''$. Reject $H_0$ if $F > F_{1,n-1,\alpha}$, where $F_{1,n-1,\alpha}$ is the $1 - \alpha$ quantile of $F_{1,n-1}$ distribution.

(b)  $* H_0: \mu \leq 0$ versus $H_1: \mu > 0$.

Solution: Similar to the harder solution of 1(c).

MLE: $\hat{\mu} = \bar{X} = 1/n \sum_{i=1}^{n} X_i$, $\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2$.

MLE under null is now a constrained maximization problem.

$$l(\mu, \sigma^2) = \log L(\mu, \sigma^2) = -n/2 \log(2\pi) - n/2 \log \sigma^2 - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2$$

First, we find the MLE of the unconstrained parameter $\sigma^2$ given a $\mu_0$.

$$\frac{\partial l}{\partial \sigma^2} = -\frac{n}{\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^{n} (X_i - \mu_0)^2$$

Its root is given by

$$\hat{\sigma}^2_0 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu_0)^2$$
Plug back in
\[
l(\mu_0, \hat{\sigma}^2_0) = -n/2 \log(2\pi) - n/2 \log \hat{\sigma}^2_0 - \frac{1}{2\hat{\sigma}^2_0} \sum_{i=1}^{n} (X_i - \mu_0)^2
\]
\[
= -n/2 \log(2\pi) - n/2 \log \left( \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu_0)^2 \right) - \frac{n}{2}
\]
is a concave function in \(\mu_0\) for any given \(\sigma^2\). In fact, its maximization problem is equivalent to the maximization problem of a quadratic function \(g(\mu_0) = \sum_{i=1}^{n} (X_i - \mu_0)^2 = n\mu^2_0 - 2n\mu_0\bar{X} + \sum_{i=1}^{n} X_i^2\). By its derivative,
\[
g'(\mu_0) = -n\mu_0 + n\bar{X} = \begin{cases} > 0 & \text{if } \mu_0 < \bar{X} \\
< 0 & \text{if } \mu_0 > \bar{X} \end{cases}
\]
Depending on the data, the MLE under null has two cases
\[
\hat{\mu}_0 = \begin{cases} 0 & \text{if } 0 < \bar{X} \\
\bar{X} & \text{if } 0 \geq \bar{X} \end{cases}
\]
Basically, we need to discuss the LR under the two cases.
Case 1: If we observe \(0 \geq \bar{X}\), then \(\hat{\mu}_0 = \hat{\mu} = \bar{X}\). GLR = \(L(\hat{\mu}_0)/L(\hat{\mu}) = L(\bar{X})/L(\bar{X}) = 1\). Rejecting GLR = 1 means that we must reject all \(\text{GLR} \leq 1\) since the GLR test has the form \(\text{GLR} \leq C\). Since \(\text{GLR} \leq 1\) with probability 1, so this means the type 1 error is always 100%, which is contradictory to the given level \(\alpha\). Therefore, we must always accept \(H_0\) if \(\hat{\mu}\) is contained in null hypothesis.
Case 2: 1: If we observe \(0 < \bar{X}\), then \(\hat{\mu}_0 = 0\).
\[
\text{GLR} = \frac{L(0, \hat{\sigma}^2_0)}{L(\hat{\mu}, \hat{\sigma}^2)} = (\frac{\hat{\sigma}^2}{\sigma^2})^{-\frac{n}{2}} \exp \left( - \frac{1}{2\hat{\sigma}^2} \sum_{i=1}^{n} X_i^2 + \frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i - \bar{X})^2 \right) = (\frac{\hat{\sigma}^2}{\sigma^2})^{-\frac{n}{2}}
\]
\[
= (\frac{n\bar{X}^2}{\sum_{i=1}^{n}(X_i - \bar{X})^2} + 1)^{-\frac{n}{2}}
\]
Like a) part, we identify GLR is the monotone decreasing function of \(D^2 = \frac{n\bar{X}^2}{\sum_{i=1}^{n}(X_i - \bar{X})^2}\). Under \(H_0\), \(F = (n-1)D^2 \sim F_{1,n-1}\). \(F\) is a monotone increasing transformation of \(\hat{\sigma}^2/\sigma^2\), so the LR test is equivalent to \(F \geq (n-1) \cdot (C' - 1) = C''\) \(\text{Reject } H_0\) if \(0 < \bar{X}\) and \(F > F_{1,n-1,\alpha}\), where \(F_{1,n-1,\alpha}\) is the \(1-\alpha\) quantile of \(F_{1,n-1}\) distribution.
Alternatively, GLR is the monotone decreasing function of \(D = \frac{\sqrt{n}\bar{X}}{\sqrt{\sum_{i=1}^{n}(X_i - \bar{X})^2}}\) if \(0 < \bar{X}\). \(T = D \cdot \sqrt{n-1} = \sqrt{n}\bar{X}/\sqrt{\sum_{i=1}^{n}(X_i - \bar{X})^2}/(n-1) \sim t_{n-1}\). \(\text{Reject } H_0\) if \(T > t_{n-1,\alpha}\), where \(t_{n-1,\alpha}\) is the \(1-\alpha\) quantile of \(t_{n-1}\) distribution.