Math 181B Worksheet Week 3

Techniques in Generalized Likelihood Ratio Test:

\( H_0 : \theta \in \Theta_0 \) versus \( H_1 : \theta \in \Theta_1 \)

- MLE: constrained (equality or inequality) optimization under null \( H_0 \). Two ways: Coordinate-wise solution; Lagrange multiplier (solve KKT equations). The solution to the constrained optimization often has case-specific form.
- Case-wise argument: generalized likelihood ratio is ALWAYS less or equal to 1; Law of Large Number argument.
- Pivotal statistics: suppress the nuisance parameter unspecified in hypotheses. \( t \)-distribution and \( F \)-distribution.

Tricks:

3. (a) First simplify the null as \( H'_0 : \sigma^2 = \sigma^2_0 \).
   
   (b) Given any fix \( \sigma^2 \), setting \( \hat{\mu} = \bar{X} \) always makes the likelihood bigger. The MLE for \( \sigma^2 \) for alternative is \( \hat{\sigma}^2 = \max \left( \sigma^2_0, n^{-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 \right) \)
   
   (c) In the case \( \hat{\sigma}^2 = \sigma^2_0 \), i.e. \( \sigma^2_0 \geq n^{-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 \), \( LR = 1 \). So you will never reject \( H_0 \).
   
   (d) If \( X \sim N(\mu, \sigma^2) \), \( \sum_{i=1}^{n} (X_i - \bar{X})^2 / \sigma^2 \sim \chi^2_{n-1} \).

4. I follow the hint from last week: \( X - Y = Z \sim N(\mu_X - \mu_Y, 2\sigma^2(1 - \rho)) = N(\nu, \tau^2) \)
   
   where \( \tau \in (0, 4\sigma^2) \). The test is \( H_0 : \nu = 0 \) vs \( H_1 : \nu \neq 0 \).
   
   (a) Given any fix \( \tau^2 \), setting \( \hat{\nu} = \bar{Z} \) always makes the likelihood bigger. Suppose \( \hat{\tau}^2_U \) is the unconstrained MLE. The constrained MLE for \( \tau^2 \) is \( \hat{\tau}^2 = \min (\hat{\tau}^2_U, 4\sigma^2) \).
   
   (b) Apply Law of Large Number to \( \hat{\tau}^2_U \to^p \tau^2 < 4\sigma^2 \). Therefore, we only need to consider the case \( \hat{\tau}^2 < 4\sigma^2 \Rightarrow \hat{\tau}^2 = \hat{\tau}^2_U \), assuming sample size is big.
   
   (c) If \( Z \sim N(\nu, \tau^2) \), \( \frac{n(\bar{Z} - \nu)^2}{(n-1) \sum_{i=1}^{n} (Z_i - \bar{Z})^2} \sim F_{1,n-1} \). The F-statistic is related to \( \hat{\tau}^2_0 / \hat{\tau}^2_1 = 1 \).

5. If \( X_1 \sim N(\mu_X, \sigma^2) \) and \( Y_1 \sim N(\mu_Y, \sigma^2) \),

\[
\frac{mn/(m+n)(\bar{X} - \mu_X - \bar{Y} + \mu_Y)^2}{\left\{ \sum_{i=1}^{m} (X_i - \bar{X})^2 + \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \right\} / (m + n - 2)} \sim F_{1,m+n-2}
\]

The F-statistic is related to \( \hat{\sigma}^2_0 / \hat{\sigma}^2_1 = 1 \).