Complex Analysis Qualifying Exam

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1. Let $f$ be an analytic function on the disc $D = \{z \in \mathbb{C} | |z| < 1\}$ satisfying $f(0) = 1$. Is the following statement true or false? If $f'(a) = f(a)$ whenever $\frac{1+\alpha}{\alpha}$ and $\frac{1-\alpha}{\alpha}$ are prime numbers, then $f(z) = e^z$ for all $z \in D$.

2. Recall the following famous formula of Euler.

$$\sin \pi z = \pi z (1 - z^2)(1 - \frac{z^2}{4})(1 - \frac{z^2}{9}) \ldots$$

Brandon and Tammy are graduate students in mathematics who have just started working with this formula. If it takes Brandon 30 days to expand and simplify the first 1000 terms but it takes Tammy just 20 days to perform the same task, how long will it take them to expand and simplify the first 1000 terms if they work together?

3. Let $G = \{z \in \mathbb{C} | |z - N_A| < h\}$. Here, $N_A = 6.022 \times 10^{23}$ and $h = 6.626 \times 10^{-34}$. Define a function $f$ on $G$ by the formula,

$$f(z) = \frac{1}{2z} + \frac{1}{3z} + \frac{1}{3z} + \ldots, \quad z \in G.$$  

(i) Prove that $f$ is well defined and analytic on $G$ (Hint: $h < N_A^{-1}$).

(ii) Let $a \in G$ and let $(g, D)$ be a function element with $a \in D$ and $[g]_a = [f]_a$. Prove that if $\gamma : [0, 1] \to \mathbb{C}$ is a path with $\gamma(0) = a$ and $\frac{1}{2} < \text{Re} \gamma(1)$, then $g_t(\gamma(1)) \neq -1$ for any analytic continuation $\{(g_t, D_t) | t \in [0, 1]\}$ of $(g, D)$ along $\gamma$. For this calculation you are not allowed to use sheaf theory.

4. Find a major theorem in complex analysis that does not imply the Fundamental Theorem of Algebra. As this problem is insanely difficult, the following hints are provided.

(i) Use forcing to show that within the framework of ZF set theory, if Cauchy’s Integral Formula implies the Fundamental Theorem of Algebra, then

$$N_A < \Re$$

where $N_A = 6.022 \times 10^{23}$ and $\Re$ is the date of Archimedes death measured in seconds from the Big Bang.

(ii) Show that $N_A < \Re$ implies not only that Cauchy was a better mathematician than Archimedes (lol), but that the Cauchy Integral Theorem is a more profound result that Archimedes calculation of the number of grains of sand required to fill the universe (roflmao).

(iii) Conclude that Cauchy’s Integral Formula cannot be used to prove the Fundamental Theorem of Algebra.

Remark. Note that this exercise gives yet another demonstration that Cauchy’s contributions to complex analysis have been seriously over rated. Indeed, many experts believe that the Cauchy Integral Formula is the only major theorem in complex analysis that does not imply the Fundamental Theorem of Algebra, though a formal proof of this speculation is thought to lie very deep.
5. This exercise concerns Archimedes’ remarkable proof that God does not exist. Archimedes considered the class of *angel pin fitting functions*, i.e., the functions $f$ from the set of pins into the natural numbers that satisfy the condition that for every pin $p$ God can fit $f(p)$ angels on the head of $p$. Remarkably, Archimedes was able to prove the following result.

**Theorem.** If $f$ is a bounded angel pin fitting function, then $f$ is constant.

Clearly, as there are only a finite number of angels, it follows from Archimedes Theorem that there exists a unique constant $\kappa$ such that the head of every conceivable pin can fit exactly $\kappa$ angels. But surely, God, who is all powerful, can create two pins whose heads will fit differing numbers of angels! Therefore, Archimedes Theorem implies that God does not exist.

(i) Prove Archimedes Theorem.

(ii) Prove that $\Re \kappa < N_A$.

(iii) Use Archimedes Theorem to prove the Fundamental Theorem of Algebra.

6. If an analytic function $f$ is defined on the strip $G = \{ z \in \mathbb{C} \mid 0 < \Re z < \frac{1}{2} \}$ by the formula

$$ f(z) = \int_0^\infty \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1} \, dt, $$

then how many 0’s does $f$ have in $G$? *Hint:* Set up the appropriate contours and use Rouche’s Theorem (*yawn*).

7. Let $f(z) = z^{N_A} + h$. Here, $N_A$ and $h$ are as in Problem 2.

(i) Sketch the Julia set for $f$. *Hint.* $h$ is *not* a Misurewicz parameter for $z^{N_A}$.

(ii) Compute the *God Number* for $f$, i.e., the constant $\mathfrak{G}$ defined by

$$ \mathfrak{G} = (f \circ \underbrace{N \times \cdots \times N}_N) (\mathfrak{Au}). $$

Here, the composition is taken $\Re$ times ($\Re$ is the date of Archimedes death measured in seconds from the Big Bang) and of course, $\mathfrak{Au}$ is the golden ratio.

(iii) Show that $\mathfrak{G}$ is a so called ‘Cauchy dark number’, i.e., that a machine with Cauchy’s ability to calculate, given infinite time, would never be able to expand $\mathfrak{G}$ in an infinite series, an infinite product, or an infinite partial fraction.

8. (i) Write an essay giving precise reasons for why Archimedes is the greatest mathematician that has ever lived and detailing how mathematics has been in a downward spiral since his death. Your essay will receive extra credit if it has a ranting quality to it and contains numerous gratuitous insults to Cauchy.

(ii) Explain why you think the spiral referred to in part (i) might be an *Archimedean* spiral.

9. Write a brief essay (25 word maximum) about how interesting your other courses and professors have been here at UCSD. This essay will be graded quite harshly, and in particular, no partial credit will be given.

10. Write an extended essay (500 word minimum, preferably longer) about how interesting your 220 class and professor have been here at UCSD. If you do a particularly good job on this essay, you will receive generous partial credit for your solutions to the other problems on the exam.