• In Lecture 2, we narrowed our study of graphs to "simple graphs" - graphs with finitely many vertices, and no loops or multiple edges.

• Simple graphs have many special properties. Among the most obvious is that their size (number of edges) is controlled by their order (number of vertices):

\[0 \leq |E| \leq \binom{|V|}{2} .\]

• We also made some elementary observations concerning the degree sequence of simple graphs:

(1) The total degree is twice the number of edges;

(2) The number of odd vertices is even;

(3) There exist two vertices of the same degree.
• Today, we study the notion of connectedness in simple graphs.

**Definition:** Vertices \( v, w \) in a graph \( \Gamma = (V, E) \) are connected if \( v = w \), or if there exists a walk from \( v \) to \( w \).

**Definition:** A subset \( C \subseteq V \) is said to be connected if \( v, w \in C \Rightarrow v, w \) connected.

**Definition:** A connected set \( C \subseteq V \) is said to be a connected component of \( \Gamma \) if it is maximal, in the sense that

\[ C \subseteq C' \text{ and } C' \text{ connected} \Rightarrow C = C'. \]

• A graph with only one connected component is connected.
Proposition: Let $\Gamma$ be a graph with exactly two odd vertices, $v$ and $w$. These vertices are connected.

Proof: If not, then $v, w$ belong to different connected components of $\Gamma$.

But then $C$ is a graph with exactly one vertex of odd degree; by the Handshake Lemma, this is impossible.
Proposition: Let $\Gamma = (V,E)$ be a graph. Then,

$$\Gamma \text{ connected} \Rightarrow |E| > |V|-1.$$

Proof: Let $v \in V$. Since $\Gamma$ is connected, there exists a walk from $v$ to every other vertex. The best case scenario is that each of the $|V|-1$ remaining vertices is adjacent to $v$, and this requires $|V|-1$ edges.

$\square$

• Note that the converse is false:

![Diagram](image)
Proposition: Let $\Gamma = (V, E)$ be a graph. Then

$$|E| \geq \binom{n-1}{2} + 1 \Rightarrow \Gamma \text{ connected.}$$

Proof: • Start with the empty graph on $\{1, \ldots, n\}$, partition the vertices into two sets

$$K_1 = \{1, \ldots, x\}, \quad K_2 = \{x+1, \ldots, n\}.$$  

• The maximal number of edges in a graph whose connected components are $K_1$ and $K_2$ is

$$\binom{x}{2} + \binom{n-x}{2}.$$  

• Now,

$$\binom{x}{2} + \binom{n-x}{2} = \frac{1}{2}x(x-1) + \frac{1}{2}(n-x)(n-x-1).$$
So, we want to find \( x \in \{1, \ldots, n-1\} \) which maximizes

\[
e(x) = \frac{1}{2} x(x-1) + \frac{1}{2} (n-x)(n-x-1)
\]

\[
= \frac{1}{2} x^2 - \frac{1}{2} x + \frac{1}{2} (n^2 - 2nx + x^2 - n + x)
\]

\[
= x^2 - nx + \binom{n}{2}
\]

\[
= (x - \frac{n}{2})^2 + \binom{n}{2} - \frac{n^2}{4}.
\]

Clearly, the function is maximized by taking \( x = 1 \) (or \( x = n-1 \)).

This means that a disconnected graph with maximal number of edges looks like \( K_{n-1} \cup \{\text{isolated vertex}\} \), a graph which has \( \binom{n-1}{2} \) edges.
**Proposition:** Two vertices \( v \) and \( w \) are connected iff there exists a path from \( v \) to \( w \).

**Proof:**

- Since a path is a walk, one direction is obvious.

- For the other direction, proceed by induction on the length of a walk from \( v \) to \( w \).

- Base step: A walk of length 1 is a path.

- Induction step: let \( v_0, v_1, ..., v_k \) be a walk of length \( k \) from \( v \) to \( w \). If this walk is a path, we're done. If not, there exists a repeated vertex. Let \( i \in \{0, ..., k-1\} \) be the first repeated vertex, i.e. smallest number such that \( v_i = v_j \) for some \( j \in \{i+1, ..., k\} \). Then, \( v_0, v_1, v_j, ..., v_k \) is a walk \( v \rightarrow w \) of length \( k-1 \) or less, so there is a path from \( v \) to \( w \) by induction.

- \( \square \)