MATH 154: Lecture 4

04/04/2016
From last time...

- Two vertices \(v,w\) in a graph \(\Gamma\) are connected if there exists a path from \(v\) to \(w\).

- A set \(C\) of vertices is connected if any two of its members are connected.

- A connected set \(C\) is a connected component if it is maximal, in the sense that

\[
\begin{align*}
C \subseteq C' & \implies C = C'. \\
C' \text{ connected} & \implies C = C'.
\end{align*}
\]

- A graph is connected if it consists of a single connected component.

Today:

- Attempt to meaningfully quantify "how connected" a graph is.
• Imagine you are a spy who has infiltrated a social network $\Gamma$. Your goal is to weaken this network by causing it to become increasingly disconnected.

• In order to achieve this goal, there are two things you can do: kill individuals, or sow discord.

**Definition:** Let $\Gamma = (V, E)$ be a graph.

For any $v \in V$, define $\Gamma - v$ to be the graph obtained from $\Gamma$ by deleting $v$, and all edges incident to $v$. More formally, $\Gamma - v$ is the graph with vertex set $V - \{v\}$, and edge set $E - \{ee : v \in e\}$.

For any $e \in E$, define $\Gamma - e$ to be the graph obtained from $\Gamma$ by deleting $e$, but leaving its endpoints intact. More formally, $\Gamma - e$ is the graph with vertex set $V$, and edge set $E - e$. 
• Sometimes, it will be the case that to disconnect $\Gamma$ you only have to kill a single individual, or dissolve a single relationship.

**Definition:** Let $\Gamma = (V,E)$ be a graph. A vertex $v \in V$ is said to be a cut-vertex if $\Gamma - v$ has more connected components than $\Gamma$. An edge $e \in E$ is said to be a bridge if $\Gamma - e$ has more connected components than $\Gamma$.

**Example:** Let $\Gamma$ be the star graph with vertex set $\{1,2,3,4,5\}$ and hub vertex 5:

The only cut vertex in a star graph is the hub, and deletion of the hub causes the number of connected components to jump from minimal (connected) to maximal (all vertices isolated).

Every edge in a star graph is a bridge.
Definition: Let $\Gamma = (V, E)$ be a graph. The vertex connectivity of $\Gamma$, denoted $\kappa(\Gamma)$, is the smallest number $\kappa$ such that one can either reduce $\Gamma$ to a single vertex, or cause it to become disconnected, by deleting $\kappa$ vertices.

The edge connectivity of $\Gamma$, denoted $\lambda(\Gamma)$, is the smallest number $\lambda$ such that deletion of $\lambda$ edges from $\Gamma$ can cause $\Gamma$ to become disconnected. The edge connectivity of a single-vertex graph is zero, by definition.

- Vertex connectivity and edge connectivity are two natural, but different, measures of "how connected" a graph is.

- A graph $\Gamma$ is said to be vertex $k$-connected if $\kappa(\Gamma) \geq k$. Thinking of $\Gamma$ as a social network, this means that at least $k$ individuals must be killed in order to produce a disconnected network, or a lone survivor.

- Similarly, $\Gamma$ is said to be edge $l$-connected if $\lambda(\Gamma) \geq l$; this means that at least $l$ friendships must be destroyed in order to cause the network to become disconnected.

- The inequalities $\kappa(\Gamma) \geq \kappa(\Gamma')$ and $\lambda(\Gamma) \geq \lambda(\Gamma')$ correspond to two senses in which one network $\Gamma$ may be considered "more connected" than another network $\Gamma'$. 
Example: If $\Gamma$ is a disconnected graph, then $k(\Gamma) = \lambda(\Gamma) = 0$.

Example: If $\Gamma$ is a star graph, $k(\Gamma) = \lambda(\Gamma) = 1$, because deletion of the hub vertex causes $\Gamma$ to become disconnected, and deletion of any edge causes $\Gamma$ to become disconnected.

Example: If $\Gamma = P_n$ is the path graph with vertex set $V = \{1, \ldots, n\}$ and edge set $E = \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}\}$, then $k(\Gamma) = \lambda(\Gamma) = 1$, because deletion of any vertex other than 1 or $n$ causes $\Gamma$ to become disconnected, and deletion of any edge causes $\Gamma$ to become disconnected.

Example: If $\Gamma = C_n$ is the cycle graph with vertex set $V = \{1, \ldots, n\}$ and edge set $E = \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}, \{n, 1\}\}$, then $k(\Gamma) = \lambda(\Gamma) = 2$. Indeed, it is not possible to disconnect $\Gamma$ by deleting a single vertex or edge, but deletion of any two non-adjacent vertices or non-incident edges results in a disconnected graph.

Example: If $\Gamma$ consists of two complete graphs which share a common vertex, e.g. \[ \begin{array}{c} \text{2} \\
\text{3} \\
\text{4} \\
\text{1} \\
\text{5} \end{array} \], then $k(\Gamma) = 1$ (deletion of the vertex 3), but $\lambda(\Gamma) > 1$. 
Proposition: Let $\Gamma$ be a graph on $n$ vertices. Then $k(\Gamma) = n-1$ if and only if $\Gamma$ is a complete graph.

Proof: First note that $k(\Gamma) \leq n-1$ for any graph on $n$ vertices - just kill any $n-1$ vertices, leaving a lone survivor.

- **Forward direction:** $k(\Gamma) = n-1 \Rightarrow \Gamma$ complete.
- **Prove contrapositive:** $\Gamma$ not complete $\Rightarrow k(\Gamma) < n-1$.
- Note that $\Gamma$ not complete implies $n \geq 2$, because any graph on a single vertex is complete by definition.
- If $\Gamma$ incomplete, it contains a pair of nonadjacent vertices $v,w$. Deleting all $n-2$ vertices except $v,w$ produces a disconnected graph, so $k(\Gamma) \leq n-2$.

- **Backward direction:** $\Gamma$ complete $\Rightarrow k(\Gamma) = n-1$.
- Note that deleting any vertex from a complete graph produces a complete graph with one less vertex.
- Consequently, you can't disconnect $\Gamma$ by deleting vertices; you have to keep killing until there's only one vertex left.

$\blacksquare$