Math 20A: Lecture 2
09/28/15, Section 1.6
• Start with the most basic number, 1, and double it repeatedly:

1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 5096, ...

• This is a sequence in which each number is twice as large as the preceding number:

\[
\begin{align*}
1 &= 2^0 \\
2 &= 2 \cdot 2 = 2^1 \\
4 &= 2 \cdot 2 \cdot 2 = 2^2 \\
8 &= 4 \cdot 2 = 2 \cdot 2 \cdot 2 = 2^3 \\
16 &= 8 \cdot 2 = 2 \cdot 2 \cdot 2 \cdot 2 = 2^4 \\
32 &= 16 \cdot 2 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 \\
64 &= 32 \cdot 2 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6 \\
128 &= 64 \cdot 2 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7 \\
\vdots
\end{align*}
\]

• The exponential notation

\[2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2 = 2^n\]

with \(n\) factors is a useful shorthand. The number 2 is called the "base," while \(n\) is the "exponent."
• This works just the same if you replace 2 with any other number, e.g. 10:

\[ 1, 10, 100, 1000, 10000, 100000, \ldots \]

...can conveniently be written

\[ 10^0, 10^1, 10^2, 10^3, 10^4, \ldots \]

• Exponential notation is useful for describing “big” numbers concisely:

- \#molecules in 1 gram of water \( \approx 3 \cdot 10^{22} \)
- radius of Earth \( \approx 6 \cdot 10^6 \) m
- distance Earth ↔ Moon \( \approx 4 \cdot 10^8 \) m
- distance Earth ↔ Sun \( \approx 1.5 \cdot 10^{11} \) m
- radius of observed universe \( \approx 10^{26} \) m
- mass of the Earth \( \approx 6 \cdot 10^{24} \) Kg
- age of the Earth \( \approx 5 \cdot 10^9 \) years
- age of the universe \( \approx 1.5 \cdot 10^{10} \) years
- population of Earth \( \approx 7.2 \cdot 10^9 \) souls
- average duration of a human life \( \approx 2 \cdot 10^9 \) s
• Start with 1 and halve it repeatedly:

\[ 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}, \frac{1}{512}, \frac{1}{1024}, \ldots \]

• Same thing in exponential notation:

\[ 2^0, 2^{-1}, 2^{-2}, 2^{-3}, 2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}, 2^{-9}, 2^{-10}, \ldots \]

• Some "small" numbers in scientific notation:

- Mass of a water molecule \( \approx 3 \cdot 10^{-23} \) g
- Size of a living cell \( \approx 10^{-9} \) m
- Wavelength of red light \( \approx 7 \cdot 10^{-7} \) m
- Breakdown of classical mechanics \( \approx 10^{-31} \) cm

• Note that "big" and "small" are relative terms: the distance from the Earth to the sun is both \( \approx 1.5 \cdot 10^{-4} \) light years, and \( \approx 1.5 \cdot 10^{10} \) m.
Some basic facts:

\[ 2^m \cdot 2^n = \underbrace{2 \cdot 2 \cdot \ldots \cdot 2}_{m} \cdot \underbrace{2 \cdot 2 \cdot \ldots \cdot 2}_{n} = \underbrace{2 \cdot 2 \cdot \ldots \cdot 2}_{m+n} = 2^{m+n} \]

\[ 2^m \cdot 2^{-n} = \underbrace{2 \cdot 2 \cdot \ldots \cdot 2}_{m} \cdot \underbrace{\frac{1}{2} \cdot \frac{1}{2} \cdot \ldots \cdot \frac{1}{2}}_{n} = \underbrace{2 \cdot 2 \cdot \ldots \cdot 2}_{m-n} = 2^{m-n} \]

\[ (2^m)^n = \underbrace{2 \cdot 2 \cdot \ldots \cdot 2}_{m} \cdot \underbrace{2 \cdot 2 \cdot \ldots \cdot 2}_{m} = 2^{mn} \]
• In calculus, we go further and study exponential functions:

\[ f(x) = b^x \]

where \( b \) is an arbitrary real number, and \( x \) is an arbitrary positive number.

• Each function \( f(x) \) in this family has the following properties:

1. The domain of \( f(x) \) is \( \mathbb{R} \) (you can multiply \( b \) with itself any number of times);

2. The codomain of \( f(x) \) is \( \mathbb{R}^+ \) (since \( b > 0 \), \( b^x > 0 \));

3. The range of \( f(x) \) is \( \mathbb{R}^+ \) (for any \( y > 0 \), \( \exists x > 0 \) so that \( y = b^x \));

4. \( f(x) \) is one-to-one (for any \( y > 0 \), \( \exists \) a unique \( x \) so that \( y = b^x \)).
A family of exponential functions
• Because \( f(x) = b^x \) is one-to-one, it has an inverse function, denoted \( f^{-1}(x) = \log_b(x) \).

• Since the domain and range of \( f(x) = b^x \) are \( \mathbb{R} \) and \( \mathbb{R}^+ \), respectively, the domain and range of \( f^{-1}(x) = \log_b(x) \) are the reverse of these:
  \[ \log_b : \mathbb{R}^+ \rightarrow \mathbb{R} \]

• For any \( x > 0 \), the number \( \log_b(x) \) is by definition the (unique) number such that
  \[ b^{\log_b(x)} = x, \]
  or equivalently
  \[ \log_b(b^x) = x. \]
Here’s the most basic logarithm identity:

\[ \log_b (a^x) = x \cdot \log_b (a). \]

Here’s why it’s true:

\[ b^{\log_b (a^x)} = a^x = (b^{\log_b (a)})^x = b^{x \log_b (a)}. \]
• Logarithmic functions "convert multiplication into addition":

\[ \log_b (x_1 x_2) = \log_b (x_1) + \log_b (x_2) . \]

• You need to know this identity. Here's why it's true:

\[ b^{\log_b (x_1 x_2)} = x_1 x_2 = b^{\log_b (x_1)} b^{\log_b (x_2)} = b^{\log_b (x_1) + \log_b (x_2)} . \]
And, here's the identity relating the logarithms of the same number $x$ with respect to different bases, $b_1$ and $b_2$:

$$\log_{b_1}(x) = \log_{b_2}(b_2) \log_{b_1}(x).$$

Here's why this is true:

$$b_1^{\log_{b_1}(x)} = x = b_2^{\log_{b_2}(x)} = (b_1^{\log_{b_1}(b_2)})^{\log_{b_2}(x)} = b_1^{\log_{b_1}(x) \log_{b_2}(x)}.$$
• In science and engineering, it’s common to use the base $b=10$ for exponentials and logarithms.

• In mathematics, we instead use the base $b=e \approx 2.71$.

• One possible definition of $e$ is as the unique number such that the slope of the line tangent to the graph of $f(x)=e^x$ at $(0,1)$ has slope 1.
• Consider a chain or piece of rope suspended by two points.

• The resulting curve is called a "catenary."

• What is the equation of the catenary?
• **Fact:** the catenary curve coincides with the graph of the function

\[ f(x) = \frac{e^x + e^{-x}}{2} = \cosh(x). \]

• This is not true if you replace e with another number, like 2 or 10.
Problem: Solve for $x$: $\log(x^2+1) - 3\log(x) = \log(2)$.

Solution: Eliminate the logs by exponentiating both sides of the equation:

$$\log(x^2+1) - 3\log(x) = \log(2) \quad \Rightarrow \quad e^{\log(x^2+1) - 3\log(x)} = e^{\log(2)}$$

$$\Rightarrow e^{\log(x^2+1)} \cdot e^{-3\log(x)} = e^{\log(2)}$$

$$\Rightarrow (x^2+1) \cdot x^{-3} = 2$$

$$\Rightarrow x^2 + 1 = 2x^3$$

$$\Rightarrow 2x^3 - x^2 - 1 = 0.$$

Now solve the cubic equation for $x$. By inspection, $x=1$ is a solution.

So, we need to factor

$$2x^3 - x^2 - 1 = (x-1)(ax^2 + bx + c)$$

$$= ax^3 + bx^2 + cx - ax^2 - bx - c$$

$$= ax^3 + (-a+b)x^2 + (c-b)x - c$$

Get $a=2$, $b=1$, $c=1.$
Thus $2x^3 - x^2 - 1 = 0 \Rightarrow (x-1)(2x^2 + x + 1) = 0$

Now use quadratic formula:

$$2x^2 + x + 1 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{1-4 \cdot 2 \cdot 1}}{2}.$$ 

So, no real roots — only real solution to $2x^3 - x^2 - 1 = 0$ is $x=1$. 

\[\square\]
Problem: The population of a city (in millions) at time \( t \) (in years) is projected to grow like \( P(t) = 2.4 e^{0.06t} \). According to this projection, when will the population double?

Solution: Since the population at present is \( P(0) = 2.4 \), we need to solve the equation

\[
2.4 e^{0.06t} = 4.8 \quad \Leftrightarrow \quad e^{0.06t} = 2.4.
\]

Taking logs, this becomes

\[
0.06t = \log(2.4)
\]

\[
\Rightarrow t = \frac{\log(2.4)}{0.06} \approx 14.6.
\]
Problem: Find the inverse of the function \( f(x) = e^{2x-3} \).

Solution: The function \( f(x) = e^{2x-3} \) has domain \( \mathbb{R} \) and codomain \( \mathbb{R}_+ \). It is one-to-one, and onto. Therefore, it admits an inverse \( g: \mathbb{R}_+ \rightarrow \mathbb{R} \). We have

\[
 f(g(x)) = e^{2g(x)-3} = x.
\]

Taking logs, we get

\[
2g(x)-3 = \log(x),
\]

so

\[
g(x) = \frac{1}{2} \left( \log(x) - 3 \right).
\]