MATH 20A: Lecture 3

09/30/15

Text §2.1, 2.2
• Lectures 1 and 2 were a review of (hopefully) familiar material.

• Today we start in on material which is (possibly) new to you; this subject matter can truly be classified as "calculus."

• Calculus is a mathematical apparatus built by Newton and Liebniz in order to solve two very specific problems.
**Problem 1:** Determine the equation of the tangent line to the graph of a given function $f(x)$ at a given point $(x_0, f(x_0))$.

**Problem 2:** Determine the area of the region under the graph of a given function $f(x)$, and lying between two given numbers $a < b$. 
**Problem 1**: Determine the equation of the tangent line to the graph of a given function \( f(x) \) at a given point \( (x_0, f(x_0)) \).

- In order to find the equation of a line, you need to know either a single point on the line and its slope, or two points on the line. Since we only have one point \( (x_0, f(x_0)) \), there's work to do.

- Physically, if \( f(x) \) is the location of a car on a straight road at time \( x \), then the slope of the tangent line is the speed of the car at time \( x_0 \).
Example: Find the equation of the tangent line to the graph of
\( f(x) = x^2 + 2x - 4 \) at the point \((2,4)\).

Solution: • First, \( f(2) = 2^2 + 2 \cdot 2 - 4 = 4 \), so \((2,4)\) is indeed a point on the graph of \( f(x) \) and the problem makes sense.

• Next, it will be helpful to have an idea of what \( f(x) \) looks like.

• \( f(x) \) is a parabola, and you can put it in vertex/intercept form by completing the square:

\[
f(x) = x^2 + 2x + 1 - 5 = (x + 1)^2 - 5
\]
Now, if we want to find the (horizontal) coordinates of all the points where \( f(x) \) and \( L(x) \) intersect, then we should solve the equation

\[
f(x) = L(x) \quad \Rightarrow \quad x^2 + 2x - 4 = m(x-2) + 4
\]

\[
\Rightarrow \quad x^2 + (2-m)x + (2m - 8) = 0.
\]

By the quadratic formula, the two solutions are

\[
x = \frac{m-2 + \sqrt{(2-m)^2 - 4(2m-8)}}{2} \quad \text{and} \quad x = \frac{m-2 + \sqrt{(2-m)^2 - 4(2m-8)}}{2}.
\]
• But, these two solutions must be the same, since the line is tangent to the parabola.

• This forces

\[(m - 2)^2 - 4(2m - 8) = 0 \implies (m^2 - 4m + 4) - 8m + 32 = 0\]

\[\implies m^2 - 12m + 36 = 0\]

\[\implies (m - 6)^2 = 0\]

\[\implies m = 6\]

• We've found the equation of the tangent line:

\[L(x) = 6(x - 2) + 4 = 6x - 8.\]
• The lesson to be learned from the last example is not that we succeeded; rather, it is that we can’t hope to be this smart every time.

• Namely, the strategy we used will allow us to solve Problem 1 whenever \( f(x) \) is a quadratic function, because we have a very good understanding of when a quadratic equation has a unique solution, which is the algebraic manifestation of the geometric property of tangency.

• Just try to repeat this using the function \( f(x) = \sin^2(x) + 2\sin(x) - 4 \) instead.
• Newton's approach to solving Problem 1 in a sense interpolates between the two ways in which one can "know a line" (i.e. slope and point vs. two points).

**Problem 1**: Determine the equation of the tangent line to the graph of a given function $f(x)$ at a given point $(x_0, f(x_0))$.

• Newton says: look at a nearby point on the curve, $(x_0+h, f(x_0+h))$, calculate the slope of the line through the two points $(x_0, f(x_0))$ and $(x_0+h, f(x_0+h))$, then repeat this many times for successively smaller values of $h$. 
The slope of the "secant line" $L_h(x)$ is

$$m(h) = \frac{f(x_0+h) - f(x_0)}{(x_0+h) - x_0} = \frac{f(x_0+h) - f(x_0)}{h}$$

The slope of the tangent line $L_0(x)$ is thus

$$m(0) = \frac{f(x) - f(0)}{0} = \frac{0}{0}.$$
The slope of the "secant line" $L_h(x)$ is

$$m(h) = \frac{f(x_0+h)-f(x_0)}{(x_0+h)-x_0} = \frac{f(x_0+h)-f(x_0)}{h}$$

We're going to have to understand how the "Newton quotient" $m(h)$ behaves when $h$ is very close but not equal to zero. This is the concept of a limit.
Example: Find the equation of the tangent line to the graph of 
\( f(x) = x^2 + 2x - 4 \) at the point \((2, 4)\).

Solution: • The Newton quotient for \( f(x) \) at \( x = 2 \) is

\[
m(h) = \frac{f(2+h) - f(2)}{(2+h) - 2} = \frac{(2+h)^2 + 2(2+h) - 4 - 4}{h} = \frac{h^2 + 6h}{h}.
\]

• If we plug in \( h = 0 \), we get \( m(0) = \frac{0^2 + 4 \cdot 0}{0} = \frac{0}{0} \), as we knew we would.

• But, we can simplify: \( m(h) = \frac{h(h+6)}{h} = h + 6 \).

• Now we get \( m(0) = 6 \).

• The equation of the tangent line is thus \( L_0(x) = 6(x-2) + 4 = 6x - 8 \).
**LIMIT DEFINITIONS**

**Intuitive:** A function \( f(h) \), defined in a neighbourhood of a point \( h_0 \), but not necessarily at \( h_0 \) itself, is said to "tend to \( L \) at \( h_0 \)" if its outputs \( f(h) \) become closer and closer to \( L \) as \( h \) becomes closer and closer to \( h_0 \).

**Rigorous:** A function \( f(h) \), defined in a neighbourhood of a point \( h_0 \), but not necessarily at \( h_0 \) itself, is said to "tend to \( L \) at \( h_0 \)" if for any \( \varepsilon > 0 \), no matter how small, there exists a corresponding \( \delta > 0 \) so that

\[
|h-h_0| < \delta \implies |f(h)-L| < \varepsilon.
\]
LIMIT NOTATION

\[ \lim_{h \to h_0} f(h) = L \]