Intuitive Definition: Let \( f(x) \) be a function defined in a neighbourhood of \( x_0 \in \mathbb{R} \), but not necessarily at \( x_0 \in \mathbb{R} \) itself. Then \( f(x) \) is said to approach \( y_0 \in \mathbb{R} \) as \( x \) approaches \( x_0 \), written

\[
\lim_{{x \to x_0}} f(x) = y_0,
\]

if \( f(x) \) gets closer and closer to \( y_0 \) as \( x \) gets closer and closer to \( x_0 \).

Rigorous Definition: Let \( f(x) \) be a function defined in a neighbourhood of \( x_0 \in \mathbb{R} \), but not necessarily at \( x_0 \in \mathbb{R} \) itself. Then \( f(x) \) is said to approach \( y_0 \in \mathbb{R} \) as \( x \) approaches \( x_0 \), written

\[
\lim_{{x \to x_0}} f(x) = y_0,
\]

if, for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that

\[
|x - x_0| < \delta \implies |f(x) - y_0| < \varepsilon.
\]
LIMIT LAWS: Let \( f(x) \) and \( g(x) \) be two functions defined in a nbhd of \( x_0 \), and suppose that \( \lim_{x \to x_0} f(x) = y_0 \) and \( \lim_{x \to x_0} g(x) = z_0 \).

(1) Set \( h(x) := f(x) + g(x) \). Then, \( \lim_{x \to x_0} h(x) = y_0 + z_0 \).

(2) Set \( h(x) := f(x) g(x) \). Then, \( \lim_{x \to x_0} f(x) g(x) = y_0 z_0 \).

(3) Assume \( z_0 \neq 0 \). Then, \( h(x) := \frac{f(x)}{g(x)} \) is defined in a nbhd of \( x_0 \), and
\[
\lim_{x \to x_0} h(x) = \frac{y_0}{z_0}.
\]
Proposition: Let $n$ be a positive integer, and let $f_1(x), \ldots, f_n(x)$ be functions defined in a nbhd of $x_0$. Suppose that
\[
\lim_{x \to x_0} f_1(x) = y_1, \ldots, \lim_{x \to x_0} f_n(x) = y_n,
\]
and define
\[
g(x) := f_1(x) + f_2(x) + \ldots + f_n(x) \\
h(x) := f_1(x)f_2(x) \ldots f_n(x).
\]
Then,
\[
\lim_{x \to x_0} g(x) = y_1 + y_2 + \ldots + y_n \\
\lim_{x \to x_0} h(x) = y_1 y_2 \ldots y_n.
\]
Proof: By induction on $n$.

Base Step: If $n=1$, then $g(x) = h(x) = f_1(x)$, and assertions coincide with hypotheses.

Induction Step: Suppose that the assertion is true for all $n=1,2,...,k$. It is required to prove that this forces the assertion to be true for $n=k+1$.

We have

$$g(x) = \underbrace{f_1(x) + \ldots + f_k(x)}_{a(x)} + f_{k+1}(x) = a(x) + f_{k+1}(x)$$

$$h(x) = \underbrace{f_1(x) \ldots f_k(x)}_{b(x)} f_{k+1}(x) = b(x) f_{k+1}(x).$$
We then have
\[
\lim_{x \to x_0} g(x) = \lim_{x \to x_0} \left[ a(x) + f_{k+1}(x) \right]
\]
\[
= \lim_{x \to x_0} a(x) + \lim_{x \to x_0} f_{k+1}(x) \quad \text{(sum law)}
\]
\[
= \lim_{x \to x_0} \left[ f_{1}(x) + \ldots + f_{k}(x) \right] + \lim_{x \to x_0} f_{k+1}(x) \quad \text{(def'n of } a(x)\text{)}
\]
\[
= [y_1 + \ldots + y_k] + y_{k+1} \quad \text{(induction hyp.)}
\]
\[
= y_1 + \ldots + y_k + y_{k+1}.
\]

Similarly,
\[
\lim_{x \to x_0} h(x) = \lim_{x \to x_0} b(x) f_{k+1}(x)
\]
\[
= \lim_{x \to x_0} b(x) \cdot \lim_{x \to x_0} f_{k+1}(x) \quad \text{(product law)}
\]
\[
= \lim_{x \to x_0} \left[ f_{1}(x) \ldots f_{k}(x) \right] \cdot \lim_{x \to x_0} f_{k+1}(x) \quad \text{(def'n of } b(x)\text{)}
\]
\[
= [y_1 \ldots y_k] y_{k+1} \quad \text{(induction hyp.)}
\]
\[
= y_1 \ldots y_k y_{k+1}.
\]
More Limit Laws: Let \( f(x) \) be a function defined in a nbhd of \( x_0 \), and suppose \( \lim_{x \to x_0} f(x) = y_0 \).

(5) Let \( t \in \mathbb{R} \), and put \( h(x) := tf(x) \). Then, \( \lim_{x \to x_0} h(x) = ty_0 \).

(6) Let \( p, q \in \mathbb{Z} \), \( q \neq 0 \), and put \( h(x) = [f(x)]^{p/q} \). Then, \( \lim_{x \to x_0} h(x) = y_0^{p/q} \).
Intuitive Definition: A function $f(x)$, defined on a nbhd of $x_0$ and at $x_0$ itself, is continuous at $x_0$ if you can draw its graph without lifting your pen from the page.
Rigorous Definition: A function $f(x)$, defined on a nbhd of $x_0$ and at $x_0$ itself, is continuous at $x_0$ if

$$\lim_{x \to x_0} f(x) = f(x_0)$$
Removable discontinuity: \( \lim_{{x \to x_0}} f(x) \) exists, but is not equal to \( f(x_0) \).
Jump Discontinuity: \( \lim_{x \to x_0} f(x) \) does not exist.

- **Left continuous**
- **Right continuous**
- **Neither**
**Continuity Laws:** Suppose that $f(x)$ and $g(x)$ are continuous at $x_0$.

1. Put $h(x) := f(x) + g(x)$. Then $h(x)$ is continuous at $x_0$.

2. Put $h(x) := f(x)g(x)$. Then $h(x)$ is continuous at $x_0$.

3. Suppose that $\lim_{x \to x_0} g(x) \neq 0$. Then $h(x) = \frac{f(x)}{g(x)}$ is defined and continuous in a nbhd of $x_0$.

4. Given $t \in \mathbb{R}$, put $h(x) := tf(x)$. Then $h(x)$ is continuous at $x_0$. 