MATH 20A: Lecture 6

10/07/15

§2.6, 2.8
• Two lectures ago (Friday), we briefly considered the behaviour of

\[ f(x) = \frac{\sin(x)}{x} \]

near the point \( x_0 = 0 \).

• We used the expansion \( \sin(x) = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \) to guess that

\[ \lim_{x \to 0} \frac{\sin(x)}{x} = 1. \]

• However, we haven’t “earned” the permissions to use infinite series yet, so our calculation was basically nonsense.

• Nevertheless, our nonsensical argument gave the correct answer – we’ll prove this today.
Squeeze Principle: Suppose you're trying to understand the behaviour of a function $f(x)$ in the nbhd of a point $x_0$. If you can find "upper" and "lower" functions $u(x)$ and $l(x)$ such that

$$l(x) \leq f(x) \leq u(x) \quad \text{for all } x \text{ near } x_0,$$

and if moreover

$$\lim_{x \to x_0} l(x) = \lim_{x \to x_0} u(x) = y_0,$$

then

$$\lim_{x \to x_0} f(x) = y_0$$

as well.
**Example:** Show that \( \lim_{x \to 0} x \sin \left( \frac{1}{x} \right) = 0 \).

**Solution:** • The function \( f(x) = x \sin \left( \frac{1}{x} \right) \) is not continuous at \( x = 0 \) (it’s not even defined there), so you can’t just blindly plug in \( x = 0 \) and hope to get the right answer (or any answer at all).

• This is a job for the Squeeze Principle: we need to find functions \( l(x) \) and \( u(x) \) such that

\[
l(x) \leq x \sin \left( \frac{1}{x} \right) \leq u(x) \quad \text{for all } x \approx 0 \quad \text{but } x \neq 0,
\]

and

\[
\lim_{x \to 0} l(x) = \lim_{x \to 0} u(x) = 0.
\]
• The basic observation is that the range of the sine function is bounded, in fact it's the interval \([-1,1]\). That is, \(\sin(\theta) \in [-1,1]\) for any \(\theta \in \mathbb{R}\).

• This means that \(|x \sin(\frac{1}{x})| = |x| \cdot |\sin(\frac{1}{x})| \leq |x| \cdot 1 = |x|\).

• Since \(x \sin(\frac{1}{x}) \leq |x \sin(\frac{1}{x})|\) for all \(x \in \mathbb{R}\setminus\{0\}\), simply because any number is at most as large as its absolute value, we have found our upper function:

\[ u(x) = |x|. \]

• Similarly, because \(x \sin(\frac{1}{x}) \geq -|x \sin(\frac{1}{x})|\) for all \(x \in \mathbb{R}\setminus\{0\}\), simply because \(y \geq -|y|\) for any \(y \in \mathbb{R}\), we have \(x \sin(\frac{1}{x}) \geq -|x \sin(\frac{1}{x})| = -|x| \cdot |\sin(\frac{1}{x})| \geq -|x|\). This gives us our lower function:

\[ l(x) = -|x|. \]

• We now know that \(-|x| \leq x \sin(\frac{1}{x}) \leq |x| \) for all \(x \in \mathbb{R}\setminus\{0\}\).
$u(x) = |x|$  

$f(x) = x \sin \left(\frac{1}{x}\right)$  

$\ell(x) = -|x|$
• Now, since clearly

\[ \lim_{x \to 0} -|x| = \lim_{x \to 0} |x|, \]

we have that

\[ \lim_{x \to 0} x \sin \left( \frac{1}{x} \right) = 0, \]

by the Squeeze Principle.

- □
• Now we start building our case for $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$.

• I switched from the "calculus variable" $x$ to the "geometry variable" $\Theta$ in order to prime you for the old-fashioned trigonometric nature of the argument: we’ll restrict to $\Theta$ in the range $-\frac{\pi}{2} \leq \Theta \leq \frac{\pi}{2}$, so that it can be interpreted as an angle in a right triangle.
• Suppose first that $\theta$ is positive, and draw the following picture:

![Diagram of a circle with a sector and a triangle]

• Obviously,

$$\text{Area of Little Triangle} \leq \text{Area of Pie Slice} \leq \text{Area of Big Triangle}.$$
• The area of the little triangle is $\frac{1}{2} \sin(\theta)$.
• The area of the pie slice is \((\frac{\Theta}{2\pi})\cdot\pi = \frac{1}{2}\Theta\).
The area of the big triangle is

\[
\frac{1}{2} \cdot 1 \cdot h = \frac{1}{2} \tan(\theta) = \frac{1}{2} \frac{\sin(\theta)}{\cos(\theta)}
\]
\[
\frac{1}{2} \sin(\theta) \leq \frac{1}{x} \leq \frac{1}{2} \frac{\sin(\theta)}{\cos(\theta)}
\]

\[
\sin(\theta) \leq \theta \leq \frac{\sin(\theta)}{\cos(\theta)}
\]

\[
\sin(\theta) \leq \theta \\
\frac{\sin(\theta)}{\theta} \leq 1 \\
\mathcal{U}(\theta) = 1
\]

\[
\theta \leq \frac{\sin(\theta)}{\cos(\theta)} \\
\cos(\theta) < \frac{\sin(\theta)}{\theta} \\
\mathcal{L}(\theta) = \cos(\theta)
\]
• Putting this all together, we have shown that

$$\theta \epsilon (0, \frac{\pi}{2}] \implies \cos(\theta) \leq \frac{\sin(\theta)}{\theta} \leq 1.$$ 

• What happens when \( \theta \) is replaced by \(-\theta\)? Nothing:

$$\theta \epsilon [-\frac{\pi}{2}, 0) \implies \cos(\theta) \leq \frac{\sin(\theta)}{\theta} \leq 1.$$ 

• We’ve found our lower and upper functions:

$$\theta \epsilon [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\} \implies \cos(\theta) \leq \frac{\sin(\theta)}{\theta} \leq 1$$

\( l(\theta) \) \quad \text{and} \quad \text{u}(\theta)$$
By the Squeeze Principle,

\[
\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1.
\]
**Theorem:** \( \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1. \)

**Corollary:** \( \lim_{\theta \to 0} \frac{-\cos(\theta)}{\theta} = 0. \)

**Proof:**

\[
\frac{1-\cos(\theta)}{\theta} = \frac{1-\cos(\theta)}{\theta} \cdot \frac{1+\cos(\theta)}{1+\cos(\theta)} = \frac{1-\cos^2(\theta)}{\theta(1+\cos(\theta))} = \frac{\sin \theta}{\theta} \cdot \frac{\sin \theta}{1+\cos(\theta)}.
\]

\[
\lim_{\theta \to 0} \frac{1-\cos(\theta)}{\theta} = \lim_{\theta \to 0} \left( \frac{\sin \theta}{\theta} \cdot \frac{\sin \theta}{1+\cos(\theta)} \right) = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \cdot \lim_{\theta \to 0} \frac{\sin \theta}{1+\cos(\theta)} = 1 \cdot \frac{0}{2} = 0.
\]
**Intermediate Value Theorem:** Let $f:[a,b] \rightarrow \mathbb{R}$ be continuous. Then, for every number $M$ between $f(a)$ and $f(b)$, there exists a number $c \in [a,b]$ such that $f(c) = M$.

**Proof:**
**Corollary:** Any odd-degree polynomial equation

\[ a_{2m+1}x^{2m+1} + a_{2m}x^{2m} + \ldots + a_1x + a_0 = 0 \]

has a solution.

**Proof:**

- Put \( f(x) := a_{2m+1}x^{2m+1} + a_{2m}x^{2m} + \ldots + a_1x + a_0 \).

- Since \( \lim_{x \to -\infty} f(x) = -\infty \) and \( \lim_{x \to +\infty} f(x) = +\infty \), there exists \( a \in \mathbb{R} \) so that \( f(a) < 0 \), and there exists \( b \in \mathbb{R} \) so that \( f(b) > 0 \).

- Since \( f(x) \) is continuous (on all of \( \mathbb{R} \)), we can apply IVT to the situation \( f(a) < 0 < f(b) \) to conclude that there exists \( c \in \mathbb{R} \), \( a < c < b \), with \( f(c) = 0 \).

\( \square \)