MATH 20A: Lecture 12

10/21/15

§3.6
Today we’ll compute the derivatives of the basic trig functions, 
\( \sin(x) \), \( \cos(x) \), and \( \tan(x) \).

**THEOREM:** The derivatives of the basic trigonometric functions are:

\[
\frac{d}{dx} \sin(x) = \cos(x), \quad \frac{d}{dx} \cos(x) = -\sin(x), \quad \frac{d}{dx} \tan(x) = \frac{1}{\cos^2(x)}.
\]
• Let's start with the sine function, \( \sin(x) \).

• To calculate the slope of the tangent line to the graph of \( \sin(x) \) at the point \((x, \sin(x))\), we need to determine the \( h \to 0 \) limit of the Newton quotient

\[
N(h) = \frac{\sin(x+h) - \sin(x)}{h}.
\]

• As always, the Newton quotient is the undefined object \( \frac{0}{0} \) at \( h=0 \), so we need to reformulate \( N(h) \) in order to take the limit.

• To put \( N(h) \) in a form more conducive to taking the \( h \to 0 \) limit, we use the addition formula

\[
\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b).
\]
• Using the addition formula, we get

\[ N(h) = \frac{\sin(x+h) - \sin(x)}{h} \]

\[ = \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \]

\[ = \frac{\sin(x)\cos(h) - \sin(x) + \cos(x)\sin(h)}{h} \]

\[ = \sin(x)\frac{\cos(h) - 1}{h} + \cos(x)\frac{\sin(h)}{h}. \]

• Thus,

\[ \lim_{h \to 0} N(h) = \sin(x) \lim_{h \to 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h \to 0} \frac{\sin(h)}{h} \]

\[ = \sin(x) \cdot 0 + \cos(x) \cdot 1 \]

\[ = \cos(x). \]
THEOREM

The derivative of the sine function is the cosine function:

\[ \frac{d}{dx} \sin(x) = \cos(x) \]
• Now, we differentiate \( \cos(x) \).

• Form the Newton quotient: 

\[
N(h) = \frac{\cos(x+h) - \cos(x)}{h}
\]

• Use the addition formula \( \cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b) \) to simplify \( N(h) \):

\[
N(h) = \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} = \cos(x) \frac{\cos(h) - 1}{h} - \sin(x) \frac{\sin(h)}{h}
\]

• Now take the \( h \to 0 \) limit:

\[
\lim_{h \to 0} N(h) = \cos(x) \lim_{h \to 0} \frac{\cos(h) - 1}{h} - \sin(x) \lim_{h \to 0} \frac{\sin(h)}{h} = -\sin(x).
\]
THEOREM

The derivative of the sine function is the cosine function:

\[ \frac{d}{dx} \sin(x) = \cos(x) \]

The derivative of the cosine function is the negative sine function:

\[ \frac{d}{dx} \cos(x) = -\sin(x) \]
• Moving particle interpretation.

The instantaneous horizontal velocity of the particle is
\[ \vec{h}(\theta) = (-\sin \theta, 0) \]
and the instantaneous vertical velocity is
\[ \vec{v}(\theta) = (0, \cos \theta). \]
• Now that we know how to differentiate $\sin(x)$ and $\cos(x)$, we can differentiate any rational combination of these using the product and quotient rules.

• Differentiation of the tangent function using the quotient rule:

$$
\frac{d}{dx} \tan(x) = \frac{d}{dx} \left( \frac{\sin(x)}{\cos(x)} \right)
= \frac{\frac{d}{dx} \sin(x) \cdot \cos(x) - \sin(x) \cdot \frac{d}{dx} \cos(x)}{\cos^2(x)}
= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}
= \frac{1}{\cos^2(x)}.
$$
• Let's go back to the geometric definition of the tangent function and differentiate it starting from that perspective. It is said that Newton differentiated many functions in this way.

• Note that \( \tan(\theta) \) is an odd, periodic function with fundamental domain \( -\frac{\pi}{2} < \theta < \frac{\pi}{2} \), so we don't actually lose any generality by going back to highschool trig.
When we increase $\theta$ by a small angle $h$, our picture develops some new triangles:

The side lengths $u$ and $v$ are

$$u = \cos(\theta + h) - \cos(\theta), \quad v = \sqrt{1 + \tan^2\theta} \cdot \sin(h).$$

The angle formed by the sides of lengths $u,v$ is $\theta$. 
As $h \to 0$, the new (small) triangle is approximately similar to the original triangle:

\[
\frac{v}{l} \approx \frac{w}{\tan \theta} \approx \frac{u}{\sqrt{1 + \tan^2 \theta}} \approx h
\]

Thus $\frac{u}{\sqrt{1 + \tan^2 \theta}} \approx v \Rightarrow \frac{\tan(\theta + h) - \tan(\theta)}{\sqrt{1 + \tan^2 \theta}} \approx h\sqrt{1 + \tan^2 \theta} \Rightarrow \frac{\tan(\theta + h) - \tan(\theta)}{h} \approx 1 + \tan^2 \theta$. 
As $h \to 0$, the approximation

$$\frac{\tan(\theta+h)-\tan(\theta)}{h} \approx 1 + \tan^2 \theta$$

becomes exact:

$$\lim_{h \to 0} \frac{\tan(\theta+h)-\tan(\theta)}{h} = 1 + \tan^2 \theta.$$  

Thus, we've shown that $\frac{d}{d\theta} \tan \theta = 1 + \tan^2 \theta$ using geometric reasoning.

Note that

$$1 + \tan^2 \theta = 1 + \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{\cos^2 \theta + \sin^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta},$$

so our geometric computation is consistent with our quotient rule computation.