MATH 20A: Lecture 13

10/23/15

§ 3.7
• The differentiation rules discussed so far explain how \( \frac{d}{dx} \) interacts with arithmetic operations:

\[
\frac{d}{dx} (f(x) + g(x)) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)
\]

\[
\frac{d}{dx} (f(x)g(x)) = \frac{d}{dx} f(x) \cdot g(x) + f(x) \cdot \frac{d}{dx} g(x)
\]

\[
\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{\frac{d}{dx} f(x) \cdot g(x) - f(x) \cdot \frac{d}{dx} g(x)}{g(x)^2}
\]

• There's one more operation that can be performed on functions: composition.

• The rule for differentiating a composition of two functions is called the Chain Rule.
**Chain Rule**: If \( F(x) = f(g(x)) \), then

\[
F'(x) = f'(g(x))g'(x)
\]

A little more precisely: if \( g \) is differentiable at \( x \) and \( f \) is differentiable at \( g(x) \), then \( F = f \circ g \) is differentiable at \( x \), with derivative given by the formula above.
\textbf{Proof:} • As always, you have to return to the definition of the derivative:

\[ F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}. \]

• Rewrite the Newton quotient as

\[ \frac{f(g(x+h)) - f(g(x))}{h} = \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h}. \]

• The second factor gives us \( g' \):

\[ g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}. \]
• Write the first factor as

\[
\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)} = \frac{f(g(x) + g(x+h)-g(x)) - f(g(x))}{g(x+h)-g(x)}
\]

\[
= \frac{f(g(x) + k) - f(g(x))}{k},
\]

where \( k = g(x+h)-g(x) \).

• Since \( k \to 0 \) as \( h \to 0 \), we get

\[
\lim_{h \to 0} \frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)} = \lim_{k \to 0} \frac{f(g(x) + k) - f(g(x))}{k} = f'(g(x)),
\]

as required.
Example: Differentiate $F(x) = \sin(x^2)$.

Solution: Note that $F(x) = f(g(x))$, where $f(x) = \sin(x)$ and $g(x) = x^2$.

Use the Chain rule:

$$F'(x) = f'(g(x)) \cdot g'(x) = \cos(x^2) \cdot 2x = 2x \cos(x^2).$$
Proposition: \( \frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} \).

Proof: 

\begin{itemize}
  \item Write \( g(x) = f^{-1}(x) \).
  
  \item Then \( f(g(x)) = x \). Differentiating both sides,

  \[ f'(g(x))g'(x) = 1. \]

  \item Solving for \( g'(x) \) yields

  \[ g'(x) = \frac{1}{f'(g(x))} = \frac{1}{f'(f^{-1}(x))}, \]

  as claimed.
\end{itemize}
• As an (important) application of the formula for $\frac{d}{dx}f^{-1}(x)$, we compute the derivatives of the inverse trig functions

$$\arcsin(x) = \sin^{-1}(x), \quad \arccos(x) = \cos^{-1}(x), \quad \arctan(x) = \tan^{-1}(x).$$
• Derivative of $\arcsin(x)$.

$$\frac{d}{dx} \arcsin(x) = \frac{1}{\sin'(\arcsin(x))} = \frac{1}{\cos(\arcsin(x))}.$$ 

• This can be simplified:

$$\cos(\arcsin(x)) = \sqrt{1-x^2}$$

• Conclusion:

$$\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$
• Derivative of $\arccos(x)$.
\[
\frac{d}{dx} \arccos(x) = \frac{1}{\cos'(\arccos(x))} = \frac{1}{-\sin(\arccos(x))}
\]

• This can be simplified:

\[
\sin(\arccos(x)) = \sqrt{1-x^2}
\]

• Conclusion:
\[
\frac{d}{dx} \arccos(x) = \frac{-1}{\sqrt{1-x^2}}
\]
• Derivative of \( \arctan(x) \).
\[
\frac{d}{dx} \arctan(x) = \frac{1}{\tan'(\arctan(x))} = \cos^2(\arctan(x)).
\]

• This can be simplified:
\[
\cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}}
\]

• Conclusion:
\[
\frac{d}{dx} \arctan(x) = \frac{1}{1 + x^2}
\]